NMAI 076 - Algebra 2 - spring semester 2024

Homework 2

Deadline 18.04.2024, 15:40

Exercise 1. (10 points)

1. Show that $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are not isomorphic. (Hint: the solutions of an equation $x^2 = r$, for $r \in \mathbb{Q}$, would be invariant under such isomorphism)
2. For which $r, s \in \mathbb{Z}$ are $\mathbb{Q}(\sqrt{r})$ and $\mathbb{Q}(\sqrt{s})$ isomorphic?
3. Show that all algebraic extensions of $\mathbb{R}$ of degree 2 are isomorphic.

Exercise 2. (10 points) The splitting field of a polynomial is the smallest field extensions, in which the polynomial decomposes into linear factors. Determine (a simple description of) the splitting field of $f = x^4 + 5x^2 + 5 \in \mathbb{Q}[x]$ and its degree over $\mathbb{Q}$.

Exercise 3. (10 points)

Prove or disprove the following statements:

1. Let $f \in \mathbb{C}[x]$ and let $\alpha \in \mathbb{C}$. If $f \notin \mathbb{Q}[x]$ and α is algebraic, then $f(\alpha) \neq 0$.
2. Let $f \in \mathbb{C}[x]$ and let $\alpha \in \mathbb{C}$. If α is transcendental and $f(\alpha) = 0$, then $f \notin \mathbb{Q}[x]$.
3. Let $f = x^8 + \sqrt{2}x^6 + 3 \in \mathbb{R}[x]$ and let $\alpha \in \mathbb{C}$ be such that $f(\alpha) = 0$. Then α is algebraic (over $\mathbb{Q}$).
4. Let $\mathbf{T} \leq \mathbf{S} \leq \mathbf{U}$ field extensions, such that $\mathbf{S}$ is algebraic over $\mathbf{T}$, and $\mathbf{U}$ is algebraic over $\mathbf{S}$. Then $\mathbf{U}$ is algebraic over $\mathbf{T}$.