## NMAI 076 - Algebra 2 - spring semester 2025 Homework 2

## Deadline 17.04.2025, 14:00

## Exercise 1. (10 points)

Prove or disprove the following statements:

- 1. Let  $f \in \mathbb{C}[x]$  and let  $\alpha \in \mathbb{C}$ . If  $f \notin \mathbb{Q}[x]$  and  $\alpha$  is algebraic, then  $f(\alpha) \neq 0$ .
- 2. Let  $f \in \mathbb{C}[x]$  and let  $\alpha \in \mathbb{C}$ . If  $\alpha$  is transcendental and  $f(\alpha) = 0$ , then  $f \notin \mathbb{Q}[x]$ .
- 3. Let  $f = x^8 + \sqrt{2}x^6 + 3 \in \mathbb{R}[x]$  and let  $\alpha \in \mathbb{C}$  be such that  $f(\alpha) = 0$ . Then  $\alpha$  is algebraic (over  $\mathbb{Q}$ ).
- 4. Let  $\mathbf{T} \leq \mathbf{S} \leq \mathbf{U}$  field extensions, such that  $\mathbf{S}$  is algebraic over  $\mathbf{T}$ , and  $\mathbf{U}$  is algebraic over  $\mathbf{S}$ . Then  $\mathbf{U}$  is algebraic over  $\mathbf{T}$ .
- **Exercise 2.** (10 points) The splitting field of a polynomial is the smallest field extensions, in which the polynomial decomposes into linear factors. Determine (a simple description of) the splitting field of  $f = x^4 + 5x^2 + 5 \in \mathbb{Q}[x]$  and its degree over  $\mathbb{Q}$ .
- **Exercise 3.** (10 points) Let  $p_1, \ldots, p_n$  be pairwise different primes. Show that  $[\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}) : \mathbb{Q}] = 2^n$ .