CSP lecture 24/25 – Problem Set 1

$\mathbb{A} = (A; R_1, R_2, \ldots)$ is called *a relational structure* if

- A is a set, called *domain*,
- $R_1, R_2, \ldots$ are *relations* on A, i.e. $R_i \subseteq A^{n_i}$ for some finite arity $n_i \geq 1$.

CSP($\mathbb{A}$)
Given a list of constraints $R_i(x_{i_1}, \ldots, x_{i_r})$, $R_j(x_{j_1}, \ldots, x_{j_s})$, $R_k(x_{k_1}, \ldots, x_{k_t}), \ldots$ **Decide** whether they are satisfiable.

Consider the following relations on $\{0, 1\}$:

- $C_i := \{i\}$, for $i \in \{0, 1\}$
- $R := \{(0, 0), (1, 1)\}$
- $N := \{(0, 1), (1, 0)\}$
- $S_{ij} := \{0, 1\}^2 \setminus \{(i, j)\}$, for $i, j \in \{0, 1\}$
- $H := \{0, 1\}^3 \setminus \{(1, 1, 0)\}$
- $G_1 := \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$, $G_2 := \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}$

Problem 1. Find a polynomial–time algorithm for CSP($\mathbb{A}$), where

1. $\mathbb{A} = (\{0, 1\}; R)$
2. $\mathbb{A} = (\{0, 1\}; R, C_0, C_1)$
3. $\mathbb{A} = (\{0, 1\}; S_{10})$
4. $\mathbb{A} = (\{0, 1\}; S_{10}, C_0, C_1)$
5. $\mathbb{A} = (\{0, 1\}; S_{01}, S_{10}, C_0, C_1)$
6. $\mathbb{A} = (\{0, 1\}; N)$
7. $\mathbb{A} = (\{0, 1\}; R, N, C_0, C_1)$
8. $\mathbb{A} = (\{0, 1\}; R, N, C_0, C_1, S_{00}, S_{01}, S_{10}, S_{11})$
9. $\mathbb{A} = (\{0, 1\}; \text{all unary and binary relations})$

Problem 2. Find a polynomial–time algorithm for CSP($\{0, 1\}; H, C_0, C_1$).

Problem 3. Find a polynomial–time algorithm for CSP($\{0, 1\}; C_0, C_1, G_1, G_2$).

Problem 4. Find a polynomial–time algorithm for CSP($\mathbb{Q}; <$).

Problem 5. Prove that CSP($\mathbb{Q}; <$) $\neq$ CSP($\mathbb{A}$), for every finite relational structure $\mathbb{A} = (A; R)$.

CSP lecture 24/25 – Problem Set 2

The *type* of a relational structure $(A; R_1, \dots, R_s)$ is the tuple $(\mathrm{ar}(R_1), \dots, \mathrm{ar}(R_s))$, where $\mathrm{ar}(R)$ is the arity of the relation R.

Suppose the type of $\mathbb{A} = (A; R_1, \dots, R_t)$ and $\mathbb{B} = (A; S_1, \dots, S_t)$ is $(n_1, \dots, n_t)$. A mapping $\phi : A \to B$ is called *a homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if $(a_1, \dots, a_{n_i}) \in R_i \Rightarrow (\phi(a_1), \dots, \phi(a_{n_i})) \in S_i$ for every i. If such a homomorphism exists we write $\mathbb{A} \to \mathbb{B}$. A homomorphism $\mathbb{A} \to \mathbb{A}$ is *an endomorphism*, a bijective endomorphism is *an automorphism*.

$\mathrm{Hom}(\mathbb{A})$
Given a finite relational structure $\mathbb{X}$ of the same type as $\mathbb{A}$. **Decide** whether $\mathbb{X} \to \mathbb{A}$.

Problem 1. Find a polynomial algorithm for $\mathrm{Hom}(\mathbb{A})$ where

1. $\mathbb{A} = (\{0,1\}; N)$ (notation is from the 1st problem set)
2. $\mathbb{A} = (\{0,1\}; N, C_0, C_1)$ (notation is from the 1st problem set)
3. $\mathbb{A} = (\{0,1\}; S_{00}, S_{11})$ (notation is from the 1st problem set)

Recall that a decision problem $\mathcal{P}_1$ is *polynomially reducible* to $\mathcal{P}_2$ if there exists a polynomial-time algorithm that transforms an input I of $\mathcal{P}_1$ to an input $r(I)$ of $\mathcal{P}_2$ so that I is a Yes-instance iff $r(I)$ is a Yes-instance. In such a case, we write $\mathcal{P}_1 \leq_P \mathcal{P}_2$. When $\mathcal{P}_1 \leq_P \mathcal{P}_2 \leq_P \mathcal{P}_1$, we write $\mathrm{CSP}(\mathbb{A}) \sim_P \mathrm{CSP}(\mathbb{B})$ and say that the two problems are *polynomially equivalent.*

Problem 2. $\mathbb{A} = (\{0,1,2\}; N)$, where $N = \{0,1,2\}^2 \setminus \{(0,0),(1,1),(2,2)\}$. Prove that $\mathrm{CSP}(\mathbb{A})$ is polynomially equivalent to $\mathrm{Hom}(\mathbb{A})$.

Problem 3. $\mathbb{A}$ is a relational structure. Prove that $\mathrm{CSP}(\mathbb{A})$ is polynomially equivalent to $\mathrm{Hom}(\mathbb{A})$.

Observe that if $\mathrm{CSP}(\mathbb{A}) \leq_P \mathrm{CSP}(\mathbb{B})$ and $\mathrm{CSP}(\mathbb{B})$ is in P (i.e., solvable in polynomial time), then $\mathrm{CSP}(\mathbb{A})$ is in P. Similarly, if $\mathrm{CSP}(\mathbb{A}) \leq_P \mathrm{CSP}(\mathbb{B})$ and $\mathrm{CSP}(\mathbb{A})$ is NP–complete, then $\mathrm{CSP}(\mathbb{B})$ is NP–complete.

Problem 4. Prove that $\mathrm{CSP}(\mathbb{A}) \sim_P \mathrm{CSP}(\mathbb{B})$, where

- $\mathbb{A} = (\{0,1,2\}; C_0, C_1, Q)$, where

$$C_0 = \{0\}, C_1 = \{1\}, Q = \{000, 110, 120, 210, 101, 102, 201, 202, 011, 012, 021\}$$

  (Q is a ternary relation, we omit the commas and parentheses, eg. 110 stands for (1,1,0).)

- $\mathbb{B} = (\{0,1\}; C_0, C_1, G_1)$ (where the notation is from the 1st problem set).

Hint: use homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$.

Problem 5. Prove that for each finite relational structure $\mathbb{A}$ there exists a relational structure $\mathbb{B}$ such that

- there exists a homomorphism $\mathbb{A} \to \mathbb{B}$ and a homomorphism $\mathbb{B} \to \mathbb{A}$, and
- $\mathbb{B}$ is a *core*, that is, each endomorphism of $\mathbb{B}$ is an automorphism.

Problem 5.1. Deduce that we can WLOG concentrate on CSPs over cores.

Problem 5.2. Prove that such a core is unique up to isomorphism.

Problem 5.3. Find a relational structure $\mathbb{A}$ such that every structure $\mathbb{B}$ with homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$ is *not* a core. Hint: $\mathbb{A}$ can be taken to be a directed graph.

Problem 6. Suppose

- $\mathbb{A} = (A; R_1, R_2, R_4)$ is a relational structure, where each R_i is an i-ary relation.
- E is the equality relation, i.e. $E = \{(a, a) : a \in A\}$
- S is the ternary relation on A defined by

$$S(x, y, z) = R_1(x) \wedge R_2(x, z) \wedge R_4(y, z, y, x)$$

- T is the binary relation defined by $T(x, y) = (\exists z \in A)\ S(x, y, z)$

Prove that

1. $\mathrm{CSP}(A; R_1, R_2, R_4, E) \leq_P \mathrm{CSP}(\mathbb{A})$
2. $\mathrm{CSP}(A; R_1, R_2, R_4, E, S) \leq_P \mathrm{CSP}(\mathbb{A})$
3. $\mathrm{CSP}(A; R_1, R_2, R_4, E, S, T) \leq_P \mathrm{CSP}(\mathbb{A})$

Problem 6.1. Try to formulate a general theorem covering these particular cases.

Problem 7. Prove that

1. $\mathrm{CSP}(\{0, 1, 2\}; C_0, C_1, N) \sim_P \mathrm{CSP}(\{0, 1, 2\}; C_0, C_1, C_2, N)$
2. $\mathrm{CSP}(\{0, 1, 2\}; N) \sim_P \mathrm{CSP}(\{0, 1, 2\}; N')$
3. $\mathrm{CSP}(\{0, 1\}; C_0, C_1, R) \sim_P \mathrm{CSP}(\{0, 1\}; R')$

where

$$N = \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\} \qquad N' = \{0, 1, 2\}^3 \setminus \{(0, 0, 0), (1, 1, 1), (2, 2, 2)\}$$
$$R = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} \qquad R' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

Hint: try to use the general theorem from Problem 6.1.

Problem 8. Prove that $\mathrm{CSP}(\mathbb{A}), \mathrm{CSP}(\mathbb{B})$ and $\mathrm{CSP}(\mathbb{C})$ are polynomially equivalent, where

$$\mathbb{A} = (\{0, 1, 2\}; C_0, C_1, C_2, N), \quad N = \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\}$$
$$\mathbb{B} = (\{0, 1\}; S_{000}, S_{001}, S_{011}, S_{111}), \quad S_{ijk} = \{0, 1\}^3 \setminus \{(i, j, k)\}$$
$$\mathbb{C} = (\{0, 1\}; C_0, C_1, R), \quad R = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$$

Problem 9. Prove that $\mathrm{CSP}(\mathbb{A}) \sim_P \mathrm{CSP}(\{0, 1, 2\}; N)$, where $\mathbb{A}, N$ are from the previous problem.

Problem 10. For each finite relational structure $\mathbb{A}$, find an input of $\mathrm{CSP}(\mathbb{A})$ whose solutions precisely correspond to endomorphisms of $\mathbb{A}$.

Problem 11. Let $\mathbb{A}$ be a finite *core* and let $\mathbb{B}$ be the relational structure formed from $\mathbb{A}$ by adding all the unary relations $C_a = \{a\}$, $a \in A$. Prove that $\mathrm{CSP}(\mathbb{A}) \sim_P \mathrm{CSP}(\mathbb{B})$.

Problem 12. Let $\mathbb{A}$ be a finite relational structure such that $\mathrm{CSP}(\mathbb{A})$ is in P. Prove that there is a polynomial–time algorithm for finding a solution of $\mathrm{CSP}(\mathbb{A})$.

CSP lecture 24/25 – Problem Set 3

An *n-ary operation* on a set A is a mapping $A^n \to A$. *The n-ary projection onto the i-th coordinate* (on a set A) is the operation π_i^n defined by $\pi_i^n(a_1, \dots, a_n) = a_i$ for any $a_1, \dots, a_n \in A$.

An n-ary operation $f : A^n \to A$ *preserves* an m-ary relation $R \subseteq A^m$ if $f(\mathbf{r}_1, \dots, \mathbf{r}_n) \in R$ (operation is applied coordinate-wise) whenever $\mathbf{r}_1, \dots, \mathbf{r}_n \in R$. In other words, for any $m \times n$ matrix whose columns are in R, f applied to the rows of this matrix gives a tuple in R. In such a situation, we also say that R is compatible with f, or R is *invariant under* f, or f is a *polymorphism* of R.

An operation $A^n \to A$ is a *polymorphism* of a relational structure $\mathbb{A} = (A; \dots)$ if it preserves all the relations in $\mathbb{A}$. The set of all polymorphisms of $\mathbb{A}$ is denoted $\mathrm{Pol}(\mathbb{A})$.

Problem 1. Observe that

1. $f : A^n \to A$ is compatible with every singleton unary relation $\{a\}$, $a \in A$, iff $f(a, \dots, a) = a$ for all $a \in A$;
2. the constant unary operation $c_a : A \to A$ (defined by $c_a(x) = a$ for any $x \in A$) is compatible with $R \subseteq A^n$ iff R contains the tuple $(a, a, \dots, a)$.

Problem 2. Let A be a set. Prove that f preserves every relation on A if and only if f is a projection.

Problem 3. Let $\mathbb{A} = (A; \dots)$ be a relational structure, $f \in \mathrm{Pol}(\mathbb{A})$ a binary polymorphism and $g \in \mathrm{Pol}(\mathbb{A})$ a ternary polymorphism. Then the 4-ary operation h defined by

$$h(x_1, x_2, x_3, x_4) = g(x_1, f(x_3, g(x_2, x_2, x_4)), x_3)$$

is a polymorphism of $\mathbb{A}$ as well. Try to formulate a general statement.

Problem 4. Find all unary and binary polymorphisms of the structure $\mathbb{A} = (\{0, 1\}; H, C_0, C_1)$ from Problem Set 1 (Problem 2 – HORN-SAT).

Problem 5. Find all unary and binary polymorphisms of the structure

$$\mathbb{A} = (\{0, 1\}; \text{all unary and binary relations})$$

from Problem Set 1 (Problem 1 – 2-SAT). Find some nice nontrivial (= not a projection) polymorphism of $\mathbb{A}$.

Problem 6. Find all unary, binary, and ternary polymorphisms of $\mathbb{A} = (\{0, 1\}; C_0, C_1, G_1, G_2)$ from Problem Set 1 (Problem 3 – LIN-EQ($\mathbb{Z}_2$)).

A relation $R \subseteq A^m$ is *pp-definable* from $\mathbb{A} = (A; \dots)$ if it can be defined from relations in $\mathbb{A}$ by a pp-formula, that is, a formula which only uses conjunction, equality, and existential quantification. A relational structure $\mathbb{B} = (B; \dots)$ is pp-definable from $\mathbb{A}$ if $A = B$ and each relation in $\mathbb{B}$ is pp-definable from $\mathbb{A}$. We also say that $\mathbb{A}$ pp-defines $\mathbb{B}$.

Problem 7. Prove that any relation pp-definable from $\mathbb{A}$ is invariant under every polymorphism of $\mathbb{A}$.

Problem 8. Find all polymorphisms of the structure $\mathbb{B}$ in Problem Set 2 (Problem 8 – 3-SAT). Hint: only projections; possible approach: (1) pp-define the four-ary relations of the form $R_{a,b,c,d} = \{0, 1\}^4 \setminus \{(a, b, c, d)\}$, (2) pp-define all four-ary relations (3) similarly, pp-define every relation, (4) use Problem 2.

Problem 9. Let $\mathbb{A}$ be a finite structure. Prove that a relation invariant under every polymorphism of $\mathbb{A}$ is pp-definable from $\mathbb{A}$. Proof strategy:

(i) Denote $R = \{(c_{11}, \ldots, c_{1k}), \ldots, (c_{m1}, \ldots, c_{mk})\}$

(ii) Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a complete list of m-tuples of elements of A (i.e. $n = |A|^m$)

(iii) Prove that the relation

$$S = \{(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_n)) : f \text{ is an } m\text{-ary polymorphism}\}$$

is pp-definable from $\mathbb{A}$ (no need to use existential quantification)

(iv) Existentially quantify over all coordinates but those corresponding to $(c_{11}, \ldots, c_{m1})$, $\ldots$, $(c_{1k}, \ldots, c_{mk})$

(v) Prove that the obtained relation contains R (because of projections) and is contained in R (because of compatibility)

Problem 10. Let $\mathbb{A} = (\mathbb{Z} \times \mathbb{Z}; R, U)$, where

$$R = \{((x, y), (x', y')) \mid x = x', |y' - y| \in \{1, 2\}\}, \quad U = \{(0, 0)\}.$$

Prove that $\{(0, y) \mid y \in \mathbb{Z}\}$ is invariant under every polymorphism of $\mathbb{A}$, but that this set is not pp-definable from $\mathbb{A}$.

Problem 11. Observe that, for finite structures $\mathbb{A}$ and $\mathbb{B}$,

1. $\mathbb{A}$ pp-defines $\mathbb{B}$ iff $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$ and in such a case $\mathrm{CSP}(\mathbb{B}) \leq_P \mathrm{CSP}(\mathbb{A})$;
2. any CSP over a two–element structure is polynomially reducible to 3–SAT
3. if $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$, then the proof of Problem 9 gives an explicit pp-formulas defining relations in $\mathbb{B}$ from relations in $\mathbb{A}$.
4. In particular, for $\mathbb{B}$ and $\mathbb{C}$ as in Problem Set 2, Problem 4, we get $\mathrm{CSP}(\mathbb{C}) \leq \mathrm{CSP}(\mathbb{B})$. How large are the explicit formulas defining relations in $\mathbb{C}$ from relations in $\mathbb{B}$?

CSP lecture 24/25 – Problem Set 4

A set of operations on a set A is a *(function) clone* on A if it contains all projections and is closed under composition (as in Problem 3, Problem Set 3). A function clone on A is called *idempotent* if for every operation f in it and every $a \in A$, $f(a, a, \dots, a) = a$. For a se

Problem 1. Recall that for any relational structure $\mathbb{A}$, $\mathrm{Pol}(\mathbb{A})$ is a clone.

In this problem set, we focus on function clones on the set $A = \{0, 1\}$. We use the following notation for some special operations on $\{0, 1\}$:

- $\wedge$ the binary minimum operation
- $\vee$ the binary maximum operation
- maj the ternary majority operation defined by $maj(a, a, b) = maj(a, b, a) = maj(b, a, a) := a$ for every $a, b \in \{0, 1\}$
- min the ternary minority operation defined by $min(a, a, b) = min(a, b, a) = min(b, a, a) := b$ for every $a, b \in \{0, 1\}$

An operation $f : A^n \to A$ is called *essentially unary* if there exist i and a unary operation $\alpha : A \to A$ such that $f(x_1, \dots, x_n) = \alpha(x_i)$ for every $x_1, \dots, x_n \in A$.

Problem 2. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither $\wedge$ nor $\vee$. Show that the only binary operations in $\mathcal{A}$ are the two projections.

Problem 3. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, maj, min$. Show that the only binary and ternary operations in $\mathcal{A}$ are the projections.

Problem 4. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, maj, min$. Show that $\mathcal{A}$ contains only projections.

Hint: possible strategy

- Let $f \in \mathcal{A}$ be n-ary with $n \geq 4$.
- Assume first $f(1, 0, 0, \dots, 0) = 1$. Use the binary operation $g(x, y) := f(x, y, \dots, y)$ to show that $f(0, 1, \dots, 1) = 0$. Use ternary operations of the form $g(x, y, z) := f(w_1, w_2, \dots)$ where $w_1, w_2, \dots \in \{x, y, z\}$ to show that f is the projection onto the first coordinate.
- Deduce that if f is not a projection, then $f(x, \dots, x, y, x, \dots, x) = x$ for every x, y and every position of y.
- Assuming this and using appropriate ternary operations (similar as above) show that $f(x, \dots, x, y, y) = x$, …, etc, and derive a contradiction

Problem 5. Let $\mathcal{A}$ be a clone on $A = \{0, 1\}$ with an operation which is not essentially unary. Prove that $\mathcal{A}$ contains a constant unary operation, or at least one of the operations $\wedge, \vee, maj, min$.
Hint: try to reduce to the idempotent case

CSP lecture 24/25 – Problem Set 5

A ternary operation $m : A^3 \to A$ is called a *majority operation* if $m(a,a,b) = m(a,b,a) = m(b,a,a) = a$ for each $a, b \in A$ (note that for $|A| \le 2$ there is a unique majority operation on A, otherwise there are more of them).

Problem 1. Let $R \subseteq A^n$ be a relation compatible with a majority operation on A. Denote $\pi_{i,j}(R)$ the projection of R onto the coordinates i, j ($1 \le i, j \le n$), that is,

$$\pi_{i,j}(R) = \{(a_i, a_j) : (a_1, \dots, a_n) \in R\} \ .$$

Prove that R is determined by these binary projections, that is,

$$(a_1, \dots, a_n) \in R \text{ if and only if } (\forall i, j,\ 1 \le i, j \le n)\ (a_i, a_j) \in \pi_{i,j}(R)$$

Hint: start with $n = 3$

Problem 2. Let $\mathbb{A} = (A; \dots)$ be a relational structure with a majority polymorphism. Show that there exists a relational structure $\mathbb{B} = (A; \dots)$ which contains only binary relations such that $\mathbb{A}$ is pp-definable from $\mathbb{B}$ and $\mathbb{B}$ is pp-definable from $\mathbb{A}$. For $A = \{0, 1\}$, conclude that CSP($\mathbb{A}$) $\le_P$ 2–SAT (and thus CSP($\mathbb{A}$) is solvable in polynomial time).

Problem 2.1. Let $\mathbb{A} = (\mathbb{Z}; R_1, \dots, R_k)$, where all relations $R_1, \dots, R_k$ admit a quantifier-free definition over the relations $y < x + c$ and $y = x + c$, where $c \in \mathbb{Z}$. E.g. R can be the 4-ary relation that holds on (x, y, z, t) iff $(x > y + 1 \vee x > z - 6) \wedge (x = z \Rightarrow t = y + 1)$ holds. Suppose that the ternary median operation is a polymorphism of $\mathbb{A}$. Show that CSP($\mathbb{A}$) is solvable in polynomial time.

Problem 3. Let $\mathbb{A} = (\{0, 1\}; \dots)$ be a relational structure with polymorphism min (from Problem Set 4). Show that each n-ary relation of $\mathbb{A}$ is an affine subspace of $\mathbb{Z}_2^n$. Conclude that CSP($\mathbb{A}$) is solvable in polynomial time.

Problem 4. Let $\mathbb{A} = (\{0, 1\}; C_0, C_1, H)$ be as in Problem Set 1 (the corresponding CSP is HORN-3-SAT). For every relation $R \subseteq \{0, 1\}^n$ compatible with $\wedge$ find a pp–definition from $\mathbb{A}$.

Problem 5. Prove that for each relational structure $\mathbb{A} = (A; \dots)$ with $A = \{0, 1\}$, either CSP($\mathbb{A}$) is solvable in polynomial time or CSP($\mathbb{A}$) is NP–complete (this is *Schaefer's dichotomy theorem* (1978)). Describe the two cases in terms of polymorphisms.

CSP lecture 24/25 – Problem Set 6

An instance of CSP($\mathbb{A}$) with set of variables V is called *1-minimal* if there exists a system of subsets $P_x \subseteq A$, $x \in V$ such that for every constraint $R(x_1, \ldots, x_k)$, the projection of R onto the j-th coordinate is equal to P_{x_j}. We say the instance is *non-trivial* if none of the sets P_x is empty.

Two instances of the CSP are *equivalent* if they have the same set of solutions.

Problem 1. Devise a polynomial-time algorithm that transforms an instance of CSP($\mathbb{A}$) to an equivalent 1-minimal instance of CSP($\mathbb{B}$), where $\mathbb{B}$ is pp-definable in $\mathbb{A}$.

Recall that a *semilattice operation* on A is a binary operation s that is associative, commutative, and idempotent: that is, for all $a, b, c \in A$, the following equalities hold:

$$
\begin{aligned}
s(s(a,b),c) &= s(a,s(b,c)) \\
s(a,b) &= s(b,a) \\
s(a,a) &= a
\end{aligned}
$$

A *totally symmetric operation* on A of arity n is an operation $t\colon A^n \to A$ such that $t(a_1, \ldots, a_n) = t(b_1, \ldots, b_n)$ whenever $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$, i.e., the value of the operation only depends on the set of its arguments.

Problem 2. Give examples of semilattice operations.

Problem 2.1. Prove that every clone that contains a semilattice operation contains for every $n \geq 1$ a totally symmetric operation of arity n.

Problem 2.2. Let $\mathbb{A}$ be finite. Prove that if Pol($\mathbb{A}$) contains totally symmetric operations of all arities $n \geq 1$, then it contains a family of totally symmetric operations $s_1, s_2, \ldots$ where s_n has arity n and $s_{n+1}(x_1, x_1, x_2, \ldots, x_n) = s_n(x_1, \ldots, x_n)$ holds for all $x_1, \ldots, x_n \in A$.

Problem 3. Suppose that $\mathbb{A}$ is a finite relational structure that has totally symmetric polymorphisms of all arities $n \geq 1$. Show that every non-trivial 1-minimal instance of CSP($\mathbb{A}$) has a solution. Conclude that CSP($\mathbb{A}$) is solvable in polynomial time.
Hint: apply the totally symmetric polymorphisms to the non-empty sets P_x whose existence is guaranteed by 1-minimality.

Problem 4. Show the converse: let $\mathbb{A}$ be finite and suppose that every non-trivial 1-minimal instance of $\mathbb{A}$ has a solution. Prove that Pol($\mathbb{A}$) contains totally symmetric polymorphisms of all arities $n \geq 1$.
Hint: Build an instance of CSP($\mathbb{A}$) whose variables are non-empty subsets of $\mathbb{A}$, and whose solutions define totally symmetric polymorphisms of $\mathbb{A}$. Show that an equivalent 1-minimal instance is non-trivial.

An instance of a CSP with variables $V = \{x_1, \ldots, x_n\}$ over the set A is called *simple* $(2,3)$-*minimal* if it satisfies all the following conditions:

- For each $1 \leq i \leq n$, there is a single unary constraint $P_i(x_i)$ where $P_i \subseteq A$,
- For each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, there is a single binary constraint $P_{i,j}(x_i, x_j)$, where $P_{i,j} \subseteq A^2$,
- $P_{i,j} = P_{j,i}^{-1}$ (i.e., $P_{i,j} = \{(b,a) \mid (a,b) \in P_{j,i}\}$),
- There are no other constraints except the ones mentioned above,
- The instance is 1-minimal: for all i, j, the restriction of $P_{i,j}$ to its first coordinate equals P_i,
- For each triple $i, j, k \in \{1, \ldots, n\}$ of distinct integers and each $(a,b) \in P_{i,j}$, there exists a $c \in P_k$ such that $(a,c) \in P_{i,k}$ and $(b,c) \in P_{j,k}$.

Problem 5. Let us represent a simple $(2, 3)$-minimal instance as a multipartite graph as follows: each variable x_i corresponds to one set whose vertices are the elements of P_i, and for every distinct i, j and $(a, b) \in P_{i,j}$, there is an edge between the corresponding vertices $a \in P_i$ and $b \in P_j$. Describe what the last two items in the definition of $(2, 3)$-minimality mean for this graph.

Problem 6. Let $\mathbb{A}$ be a finite structure and have only unary and binary relations. Devise a polynomial-time algorithm that transforms any instance of CSP($\mathbb{A}$) into an equivalent simple $(2, 3)$-minimal instance of CSP($\mathbb{B}$), where $\mathbb{B}$ is pp-definable in $\mathbb{A}$.

Problem 7. Adapt the algorithm from the previous problem for the case where $\mathbb{A}$ has relations of arbitrary arity but Pol($\mathbb{A}$) contains a majority operation.

Problem 8. Suppose that $\mathbb{A}$ has a majority polymorphism. Show that every non-trivial simple $(2, 3)$-minimal instance of CSP($\mathbb{A}$) has a solution.

Hint: if $V = \{x_1, \ldots, x_n\}$ is the set of variables and $h \colon \{x_1, \ldots, x_i\} \to A$ is an assignment that satisfies all constraints involving only the variables from $\{x_1, \ldots, x_i\}$, show that h can be extended to an assignment $h' \colon \{x_1, \ldots, x_i, x_{i+1}\} \to A$ that satisfies all the constraints involving only the variables from $\{x_1, \ldots, x_i, x_{i+1}\}$. Conclude that CSP($\mathbb{A}$) is solvable in polynomial time.

Remark 1. It is also possible to characterize the property "Every non-trivial $(2, 3)$-minimal instance of CSP($\mathbb{A}$) has a solution" in terms of Pol($\mathbb{A}$), although the proof is beyond the scope of the course: the property is equivalent to Pol($\mathbb{A}$) containing for all $n \geq 3$ an operation w of arity n that satisfies

$$w(x, y, \ldots, y) = w(y, x, y, \ldots, y) = \cdots = w(y, \ldots, y, x).$$

CSP lecture 24/25 – Problem Set 7

We assume throughout this sheet that every set is finite. A *Maltsev operation* is an operation $m\colon A^3 \to A$ that satisfies $m(a,b,b) = m(b,b,a) = a$ for all $a, b \in A$.

Problem 1. A relation $R \subseteq A^n$ is *rectangular* if for all $i \in \{1,\dots,n\}$, all $\mathbf{a}, \mathbf{b} \in A^n$, $c, d \in A$, whenever $(a_1,\dots,a_{i-1},c,a_{i+1},\dots,a_n), (b_1,\dots,b_{i-1},c,b_{i+1},\dots,b_n), (b_1,\dots,b_{i-1},d,b_{i+1},\dots,b_n) \in R$, then $(a_1,\dots,a_{i-1},d,a_{i+1},\dots,a_n) \in R$. Show that every relation that is invariant under a Maltsev operation is rectangular.

We say that $t, t' \in A^n$ *witness* $(i,a,b) \in \{1,\dots,n\} \times A^2$ if $(t_1,\dots,t_{i-1}) = (t'_1,\dots,t'_{i-1})$ and $t_i = a, t'_i = b$. Let $R \subseteq A^n$. The *signature* of R is the set

$$\mathrm{Sig}_R := \{(i,a,b) \in [n] \times A^2 \mid \exists \mathbf{t}, \mathbf{t}' \in R \text{ that witness } (i,a,b)\}.$$

We say that $R' \subseteq R$ is a *representation* of R if $\mathrm{Sig}_{R'} = \mathrm{Sig}_R$, and that the representation is *compact* if $|R'| \le 2 \cdot |\mathrm{Sig}_R|$. Note that for compact representations $|R'| \le 2n|A|^2$ holds.

Problem 2. Observe that every R has a compact representation. Describe a concrete compact representation of A^n.

Given a subset $R \subseteq A^n$ and an operation $f\colon A^m \to A$, the *relation generated by R under f*, denoted by $\langle R \rangle_f$, is the smallest relation S containing R and that is invariant under f. For $i_1,\dots,i_m \in \{1,\dots,n\}$, let $\pi_{i_1,\dots,i_m}(R) := \{(a_{i_1},\dots,a_{i_m}) \mid (a_1,\dots,a_n) \in R\}$.

Problem 3. Suppose that R is invariant under a Maltsev operation f and that R' is a representation of R. Show that $\langle R' \rangle_f = R$.
Hint: Show that $\pi_{1,\dots,i}(\langle R' \rangle_f) = \pi_{1,\dots,i}(R)$, for all $i \in \{1,\dots,n\}$.

In the next exercises, we use the following notation:

- $R \subseteq A^n$ is invariant under a Maltsev operation f,
- $R' \subseteq R$ is a compact representation of R,
- $S \subseteq A^m$ is also a relation of small arity $m < n$ that is also invariant under f (we think of m as a fixed parameter in contrast to n).
- Let $i_1,\dots,i_m \in \{1,\dots,n\}$ and $T = \{(a_1,\dots,a_n) \in R \mid (a_{i_1},\dots,a_{i_m}) \in S\}$.

Problem 4. Describe an algorithm that takes $R', (i_1,\dots,i_m), S$ as input, and returns an element of T (or 'False' if $T = \emptyset$). The running time should be polynomial in n (and $|A|^m$).
Hint: apply the Maltsev operation to R' until the projection on the coordinates $(i_1,\dots,i_m)$ stabilizes.

Problem 5. Describe an algorithm that takes R' and a constant $c \in A$ as input, and returns a compact representation of $R|_c := \{(a_1,\dots,a_n) \in R \mid a_1 = c\}$ in time polynomial in $|R'|$ and n.
Hint: given any $(i,a,b) \in \mathrm{Sig}_R$, use Problem 4 to decide whether (i,a,b) is in $\mathrm{Sig}_{R|_c}$.
Note that by iterating the algorithm, one can also compute a compact representation of

$$R|_{c_1,\dots,c_m} = \{(a_1,\dots,a_n) \in R \mid a_1 = c_1,\dots,a_m = c_m\}.$$

Problem 6. Describe an algorithm that takes R' and S as input, and returns a compact representation of T in time polynomial in n (and $|A|^m$).
Hint: simply describe the necessary and sufficient conditions for a given $(i,a,b) \in \mathrm{Sig}_R$ to be in Sig_T, and use the previous two algorithms to check those conditions.

Problem 7. Prove that if $\mathbb{A}$ is a finite relational structure such that $\mathrm{Pol}(\mathbb{A})$ contains a Maltsev polymorphism, then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time.

CSP lecture 24/25 – Problem Set 8

Given an equivalence relation $\sim$ on a set V and $v \in V$, we denote by $v/_\sim := \{w \in V \mid v \sim w\}$ the equivalence class of v. Recall that given a relational structure $\mathbb{G}$ and an equivalence relation $\sim$ on the domain of $\mathbb{G}$, the structure $\mathbb{G}/_\sim$ is the structure with same signature as $\mathbb{G}$, whose domain is the set of $\sim$-equivalence classes, and where for every k-ary relation R in the signature, we have

$$(v_1/_\sim, \dots, v_k/_\sim) \in R^{\mathbb{G}/\sim} \Leftrightarrow \exists w_1, \dots, w_k \text{ s.t. } w_1 \sim v_1, \dots, w_k \sim v_k \text{ and } (w_1, \dots, w_k) \in R^{\mathbb{G}}$$

Definition. Let $\mathbb{A}, \mathbb{B}$ be relational structure. We say that $\mathbb{B}$ has a *pp-interpretation* in $\mathbb{A}$ if $\mathbb{B}$ is isomorphic to a structure of the form $(S; R_1, \dots, R_k)/_\sim$, where:

- $S \subseteq A^n$ is pp-definable in $\mathbb{A}$,
- $\sim \subseteq S^2$ is an equivalence relation that is pp-definable in $\mathbb{A}$, as a relation of arity $2n$, i.e. there exists a pp-formula $\phi(x_1, \dots, x_n, y_1, \dots, y_n)$ such that for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in S$,

$$(a_1, \dots, a_n) \sim (b_1, \dots, b_n) \Leftrightarrow \mathbb{A} \models \phi(a_1, \dots, a_n, b_1, \dots, b_n)$$

- Similarly, for every $R_i \subseteq S^m$, there is a pp-formula $\psi_i(x_{1,1}, \dots, x_{1,n}, \dots, x_{m,1}, \dots, x_{m,n})$ with mn free variables such that

$$(\mathbf{a}_1, \dots, \mathbf{a}_m) \in R_i \Leftrightarrow \mathbb{A} \models \psi_i(\mathbf{a}_1, \dots, \mathbf{a}_m)$$

Problem 0. Show that if $\mathbb{B}$ has a pp-interpretation in $\mathbb{A}$, then CSP($\mathbb{B}$) reduces to CSP($\mathbb{A}$). Observe that if $\mathbb{C}$ has a pp-interpretation in $\mathbb{B}$ and $\mathbb{B}$ has a pp-interpretation in $\mathbb{A}$, then $\mathbb{C}$ has a pp-interpretation in $\mathbb{A}$. Hint: See Problems 3 and 4 from Problem Set 2.

The goal of this problem sheet is to show the following:

Theorem. *Let $\mathbb{G} = (\{v_1, \dots, v_n\}; E)$ be an undirected graph without loops and containing a triangle. Then $\mathbb{K}_3$ has a pp-interpretation in $(\mathbb{G}, \{v_1\}, \dots, \{v_n\})$, the relational structure obtained by expanding $\mathbb{G}$ by a unary relation for every vertex of $\mathbb{G}$.*

We prove the theorem by induction on n and $|E|$. For the base case $n \leq 3$ it clearly holds. So, for the rest of the sheet, let $n > 3$ and $\mathbb{G} = (V; E)$ with $V = \{v_1, \dots, v_n\}$ be an undirected, loopless graph containing a triangle. Our goal is to prove that $\mathbb{K}_3$ has a pp-interpretation in $(\mathbb{G}, \{v_1\}, \dots, \{v_n\})$ under the induction assumption that the theorem holds for every graph with $< n$ vertices, and every graph with n vertices and $< |E|$ edges.

Problem 1. Suppose that one of the conditions below is satisfied. Show that in every case, $(\mathbb{G}, \{v_1\}, \dots, \{v_n\})$ pp-interprets a proper subgraph $\mathbb{H} = (W; F)$ (i.e. $W \subseteq V, F \subseteq E$, and at least one of the inclusions is proper) that contains a triangle.

a) $\mathbb{G}$ is unconnected,

b) $\mathbb{G}$ contains a complete graph on 4 vertices,

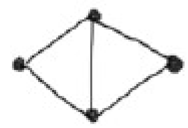
c) Some vertex v_i does not belong to a triangle,

d) Some edge of $\mathbb{G}$ does not belong to a triangle.

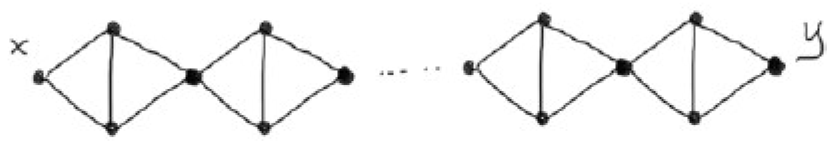
Conclude that if any of the condition holds, $\mathbb{K}_3$ has a pp-interpretation in $(\mathbb{G}, \{v_1\}, \dots, \{v_n\})$.

We assume from here on out that a)-d) do not hold for $\mathbb{G}$.

Problem 2. The diamond is the following graph:



Let $x \sim y$ be the relation that relates x and y iff they are connected by a chain of diamonds:



Show that $\sim$ is an equivalence relation that has a pp-definition in $\mathbb{G}$.

Problem 3. Suppose that the following condition holds:

e) some edge of $\mathbb{G}$ belongs to two triangles.

In particular, $\mathbb{G}$ contains a diamond and $\sim$ from Problem 2 contains a pair (x, y) with $x \neq y$.

- Show that if there is an edge (x, y) in $\mathbb{G}$ with $x \sim y$, then $(\mathbb{G}, \{v_1\}, \ldots, \{v_n\})$ pp-interprets a proper subgraph containing a triangle, thus $\mathbb{K}_3$ has a pp-interpretation in $(\mathbb{G}, \{v_1\}, \ldots, \{v_n\})$.
- Next suppose that $x \sim y$ implies that (x, y) is not an edge. What does this imply for $(\mathbb{G}, \{v_1\}, \ldots, \{v_n\})/_\sim$? Conclude that $\mathbb{K}_3$ has a pp-interpretation in $(\mathbb{G}, \{v_1\}, \ldots, \{v_n\})$.

Hint: for the first part, consider the shortest chain of diamonds connecting an edge (x, y), and find a pp-definition of a proper subset of V containing a triangle.

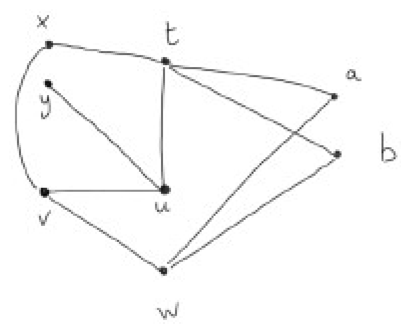
So, from here on, we can also assume that condition e) fails, i.e., every edge of $\mathbb{G}$ belongs to a unique triangle. The next goal is to show that some power of $\mathbb{K}_3$ has a pp-interpretation in $(\mathbb{G}, \{v_1\}, \ldots, \{v_n\})$. For $k \geq 1$, let $\mathbb{P}_k := (\mathbb{K}_3)^k$ be the k-th power of $\mathbb{K}_3$, whose universe is $\{1, 2, 3\}^k$ and whose edges are of the form $(\mathbf{a}, \mathbf{b})$ where for all $i \in \{1, \ldots, k\}, a_i \neq b_i$.

Problem 4. Let $h : \mathbb{P}_k \to \mathbb{G}$ be a homomorphism. Show that there is a set $I \subseteq \{1, \ldots, k\}$ such that for all $\mathbf{x}, \mathbf{y} \in \{1, 2, 3\}^k$

$$h(\mathbf{x}) = h(\mathbf{y}) \Leftrightarrow \forall i \in I, x_i = y_i.$$

Conclude that the subgraph of $\mathbb{G}$ induced by the range of h is isomorphic to $\mathbb{P}_m$, where $m = |I|$. The following strategy can be used:

- Let $I \subseteq \{1, \ldots, k\}$ be maximal such that $h(\mathbf{x}) = h(\mathbf{y})$ implies $x_i = y_i$ for all $i \in I$.
- Let $j \in \{1, \ldots, k\} \backslash I$ and let $\mathbf{a}, \mathbf{b}$ tuples that agree on all coordinates except $a_j \neq b_j$. We are going to show that $h(\mathbf{a}) = h(\mathbf{b})$.
- By maximality of I, there exist $\mathbf{x}, \mathbf{y}$ such that $h(\mathbf{x}) = h(\mathbf{y})$ but $x_j \neq y_j$. Wlog. $x_j \in \{a_j, b_j\}$.
- Show that the following graph is a (non-induced) subgraph of $\mathbb{P}_k$ (i.e., find witnesses for the vertices t, u, v, w) and use it to conclude that $h(\mathbf{a}) = h(\mathbf{b})$:



- Finally, conclude that if $a_i = b_i$ for all $i \in I$, then $h(\mathbf{a}) = h(\mathbf{b})$.

Let k be maximal such that $\mathbb{P}_k$ is isomorphic to an induced subgraph of $\mathbb{G}$ (note that $k \geq 1$ is well-defined since $\mathbb{G}$ contains a triangle). By abuse of notation, we consider $\mathbb{P}_k$ itself to be an induced subgraph of $\mathbb{G}$.

Problem 5. Show that the vertex set of $\mathbb{P}_k$ is pp-definable in $(\mathbb{G}, \{v_1\}, \ldots, \{v_n\})$.
Hint: This is equivalent to showing that for every idempotent polymorphism f of $\mathbb{G}$, the vertex set of $\mathbb{P}_k$ is invariant under f. Observe that f induces a homomorphism $(\mathbb{P}_k)^n = \mathbb{P}_{nk} \to \mathbb{G}$, where n is the arity of f.

Problem 6. To conclude the proof of the theorem, show that for all $k \geq 1$, $\mathbb{K}_3$ has a pp-interpretation in the expansion of $\mathbb{P}_k$ by all unary constant relations.
Hint: show that the equivalence relation $\mathbf{x} \sim \mathbf{y} :\Leftrightarrow x_1 = y_1$ is pp-definable in the expansion of $\mathbb{P}_k$ by all unary constant relations. There are two approaches, either by finding a concrete pp-definition, or by showing that $\sim$ is preserved under every idempotent polymorphism of $\mathbb{P}_k$.

Problem 7. Show the following corollary (Hell-Nešetřil, 1990): let $\mathbb{G} = (V; E)$ be a finite undirected graph without loops. Then CSP($\mathbb{G}$) is in P if $\mathbb{G}$ is bipartite, and CSP($\mathbb{G}$) is NP-complete otherwise.
Hint: if $\mathbb{G} = (V; E)$ is not bipartite, it has a cycle of length $2\ell + 1$ for some ℓ. Take ℓ minimal. Consider the graph $\mathbb{H}$ on V where (x, y) is an edge iff in $\mathbb{G}$ there is a walk of length $2\ell - 1$ between x and y. What can be said about $\mathbb{H}$?

CSP lecture 24/25 – Problem Set 9

All sets in this sheet are assumed finite. Clones are idempotent. (These assumptions are sometimes not necessary.)

A relation $R \subseteq A^2$ is *subdirect*, written $R \subseteq_{\text{sd}} A^2$, if its projection to each of the two coordinates is equal to A. A relation $R \subseteq A^2$ is *linked* if it is subdirect and, for every pair $a, a' \in A$, there is a "fence" from a to a', i.e. there are elements $a = a_0, b_0, a_1, b_1, \dots, b_{n-1}, a_n = a' \in A$ such that $R(a_0, b_0)$, $R(a_1, b_0)$, $R(a_1, b_1)$, $R(a_2, b_1)$, $\dots$, $R(a_n, b_{n-1})$ holds.

Problem 1. Suppose that $\mathbb{G} = (V; E)$ is a connected undirected graph. Show that $E \subseteq V^2$ is linked iff $\mathbb{G}$ is non-bipartite.

Problem 2. Let $R \subseteq A^2$. Show that there exists a largest $B \subseteq A$ (w.r.t. inclusion) such that $R \cap (B \times B) \subseteq_{\text{sd}} B^2$ and show that this B is pp-definable from R. Let's call this B the "subdirect part" of R. Show that the subdirect part of R is nonempty iff R contains a directed cycle.

Let $f : A^n \to A$ and $B \subseteq A$. We say that B *absorbs* A *with respect to* f, and write $B \triangleleft_f A$, if $f(a_1, \dots, a_n) \in B$ whenever all the a_i but at most one are in B. For a clone $\mathcal{A}$ on A, we say that B is an *absorbing subuniverse of* $\mathcal{A}$ *(with respect to* f*)*, written $B \triangleleft_f \mathcal{A}$, if B is invariant under $\mathcal{A}$, $f \in \mathcal{A}$ and $B \triangleleft_f A$. We write $B \triangleleft \mathcal{A}$ if there exists a $f \in \mathcal{A}$ such that $B \triangleleft_f \mathcal{A}$.

Problem 3. Consider the important idempotent clones on $\{0, 1\}$ (generated by the binary minimum/maximum, majority, minority). What are the absorbing subuniverses?

Problem 4. Let $\mathcal{A}$ be a clone. Suppose that $R \subseteq_{\text{sd}} A^2$ is invariant under $\mathcal{A}$ and $B, C \triangleleft_f \mathcal{A}$. Show that $B \cap C \triangleleft_f \mathcal{A}$, that $B + R := \{c : \exists b\ (b, c) \in R\} \triangleleft_f \mathcal{A}$, and that the "subdirect part" of $B \cap (R \times R)$ absorbs $\mathcal{A}$ with respect to f, as well.
Side note: Observe that for $B \triangleleft_f \mathcal{A}$ and $C \triangleleft_g \mathcal{A}$, there is a common $h \in \mathcal{A}$ such that $B, C \triangleleft_h \mathcal{A}$.
Hint: use star composition defined below.

Problem 5. Let $\mathcal{A}$ be a clone. Suppose that $R \subseteq_{\text{sd}} A^2$ is linked and invariant under $\mathcal{A}$, $B \triangleleft_f \mathcal{A}$, and $S := R \cap (B \times B) \subseteq_{\text{sd}} B^2$. Show that S is linked.

Problem 6. Let $R \subseteq A^2$ be linked and invariant under $\mathcal{A}$ and let $B \triangleleft \mathcal{A}$ be nontrivial (i.e., $\emptyset \neq B \subsetneq A$). Show that there exists a nontrivial $C \subsetneq A$ invariant under $\mathcal{A}$ such that $S := R \cap (C \times C) \subseteq_{\text{sd}} C^2$ and S is linked.
Hint: Find $B' \triangleleft \mathcal{A}$ such that $R \cap (B' \times B')$ has a nonempty subdirect part.

Let $f : A^n \to A$ and $\alpha : [n] \to [m]$. The operation $f^\alpha : A^m \to A$ defined by $f^\alpha(a_1, \dots, a_m) = f(a_{\alpha(1)}, a_{\alpha(2)}, \dots, a_{\alpha(n)})$ is called *a minor* of f. For two clones $\mathcal{A}$, $\mathcal{B}$, an arity preserving mapping $\xi : \mathcal{A} \to \mathcal{B}$ is a *minion homomorphism* if it preserves minors, i.e., $\xi(f^\alpha) = [\xi(f)]^\alpha$ (for every n, n-ary $f \in \mathcal{A}$, and every $\alpha : [n] \to [m]$).
A clone is called *Taylor* if it is idempotent and there exists no homomorphism from ξ to the clone of projections (say, on a two-element set).

Remark: There exists a minion homomorphism $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Pol}(\mathbb{B})$ iff $\mathbb{A}$ *pp-constructs* $\mathbb{B}$, i.e. $\mathbb{B}$ can be obtained from $\mathbb{A}$ by homomorphic equivalence and pp-interpretations. So $\mathrm{Pol}(\mathbb{A})$ is not Taylor iff $\mathbb{A}$ pp-constructs all finite structures. As a consequence, $\mathrm{CSP}(\mathbb{A})$ is NP-complete if $\mathrm{Pol}(\mathbb{A})$ is not Taylor.

A subset $B \subseteq A$ is called a *projective subuniverse* of $\mathcal{A}$ if for every $f \in \mathcal{A}$ there exists a coordinate i such that $f(a_1, \dots, a_n) \in B$ whenever $a_i \in B$.

Problem 7. Let B be a projective subuniverse of $\mathcal{A}$. Show that $B \triangleleft_g \mathcal{A}$ (where g can be taken binary) or $\mathcal{A}$ is not Taylor.

Hint: Show that if for each f the coordinate i (from the definition of projective subuniverse) is unique, then we get a minion homomorphism to projections. Otherwise, a binary minor of an operation f with non-unique i gives binary absorption.

Problem 8. Suppose that $\mathcal{A}$ has no nontrivial projective subuniverses. Show that $\mathcal{A}$ contains a *transitive operation*, i.e., $f \in \mathcal{A}$ such that for every coordinate i and every $a, b \in A$, there exists $(a_1, \dots, a_n) \in A^n$ such that $a_i = a$ and $t(a_1, \dots, a_n) = b$.

Hint: try to make $t(A, A, \dots, a, A, A, \dots)$ as large as possible; use the "star-product" of operations, where for n-ary f and m-ary g, we define nm-ary $f \star g$ by

$$f \star g(a_1, \dots, a_{nm}) = f(g(a_1, \dots, a_m), g(a_{m+1}, \dots, a_{2m}), \dots, g(a_{m(n-1)+1} \dots, a_{mn})).$$

The *left center* of $R \subseteq A^2$ is the set $\{a : \forall b \in A \ (a, b) \in R\}$.

Problem 9. Suppose that $R \subseteq_{\text{sd}} A^2$ is invariant under a transitive operation $f : A^n \to A$ and let B be the left center of R. Show that $B \lhd_f A$.

Problem 10. Suppose that $R \subseteq_{\text{sd}} A^2$ is linked. Show that R together with the singleton unary relations $\{a\}$ pp-defines a relation $S \subseteq_{\text{sd}} A^2$, $S \neq A^2$ with a nonempty left center.

Hint: Denote by T_n the n-ary relation such that $T_n(a_1, \dots, a_n)$ iff there exists b with $R(a_1, b)$, $R(a_2, b)$, $\dots$, $R(a_n, b)$. First adjust R so that it is still proper and $T_2 = A^2$. Fixing appropriate values in an appropriate T_n gives us S.

Problem 11. Suppose that $\mathcal{A}$ is Taylor and $R \subseteq A^2$ is linked and invariant under $\mathcal{A}$. Show that there exists a nontrivial $B \lhd \mathcal{A}$. (This is the so-called *Absorption Theorem*.)

Problem 12. Suppose that $\mathcal{A}$ is Taylor and $R \subseteq A^2$ is linked and invariant under $\mathcal{A}$. Show that $(a, a) \in R$ for some $a \in A$. (This is the so-called *Loop Lemma*.) Deduce the Hell–Nešetřil dichotomy theorem for undirected graphs (Problem 7 in Problem Set 8)

CSP lecture 24/25 – Problem Set 10

In (the feasibility version of) linear programs (LPs) the task is to decide if a finite list of linear equations and inequalities is satisfiable over $\mathbb{Q}$ or not (i.e. is there a vector $\mathbf{x} \in \mathbb{Q}^n$ that satisfies inequality constraints $A\mathbf{x} \leq \mathbf{b}$ and equality constraints $E\mathbf{x} = \mathbf{f}$, for some $A \in \mathbb{Q}^{n \times k}, \mathbf{b} \in \mathbb{Q}^k$, $E \in \mathbb{Q}^{n \times l}$, and $\mathbf{f} \in \mathbb{Q}^l$?). It is a famous result in optimization that LPs can be solved in polynomial time.

Problem 1. Let $\mathbb{A}$ be a finite relational structure, and $\mathbb{X}$ be an instance of CSP($\mathbb{A}$). Then we define a linear program as follows:

- for every $x \in X$ and every value $a \in A$ we introduce a variable $\lambda_x(a)$ together with the inequality constraints $0 \leq \lambda_x(a) \leq 1$ and the equality constraints $\sum_{a \in A} \lambda_x(a) = 1$ (so, for every $x \in X$, $\lambda_x(\cdot)$ is a probability distribution on A).
- for every constraint C in $\mathbb{X}$, given by $(x_1, \ldots, x_k) \in R^{\mathbb{X}}$, and every tuple $\mathbf{a} \in R^{\mathbb{A}}$, we introduce a variable $\lambda_C(\mathbf{a})$ together with the inequality constraints $0 \leq \lambda_C(\mathbf{a}) \leq 1$ and the equality constraints $\sum_{\mathbf{a} \in R^{\mathbb{A}}} \lambda_C(\mathbf{a}) = 1$ (so $\lambda_C(\cdot)$ is a probability distribution on $R^{\mathbb{A}}$).
- Additionally we add the following compatibility condition for every constraint C (given by $(x_1, \ldots, x_k) \in R^{\mathbb{X}}$), index i and $b \in A$:

$$\sum_{\mathbf{a} \in R^{\mathbb{A}}, a_i = b} \lambda_C(\mathbf{a}) = \lambda_{x_i}(b). \tag{1}$$

The resulting LP is the *basic linear programming relaxation* $BLP_{\mathbb{A}}(\mathbb{X})$ of $\mathbb{X}$.

Discuss how solutions $h \colon \mathbb{X} \to \mathbb{A}$ correspond to $\{0,1\}$-valued solutions of $BLP_{\mathbb{A}}(\mathbb{X})$.

Problem 2. Find a tractable CSP($\mathbb{A}$) with NO-instance $\mathbb{X}$, such that the relaxation $BLP_{\mathbb{A}}(\mathbb{X})$ has a solution.
Hint: it is enough to consider $|A| = 2$.

Our goal in the following is to characterize those finite structures $\mathbb{A}$ for which $\mathbb{X} \to \mathbb{A}$ if and only if $BLP_{\mathbb{A}}(\mathbb{X})$ is solvable. For such templates $\mathbb{A}$, we say *BLP solves* CSP($\mathbb{A}$). Note that CSP($\mathbb{A}$) is in P, if it is solvable by BLP.

Problem 3. Define an (infinite) relational structure $\mathbb{A}'$, such that $BLP_{\mathbb{A}}(\mathbb{X})$ has a solution if and only if $\mathbb{X} \to \mathbb{A}'$.
Hint: The domain of $\mathbb{A}'$ consists of rational probability distributions on A. How to define the relations?

Problem 4. Show that BLP solves CSP($\mathbb{A}$) if and only if there is a homomorphism $h \colon \mathbb{A}' \to \mathbb{A}$.

Problem 5. Show that there is a homomorphism $h \colon \mathbb{A}' \to \mathbb{A}$ if and only if $\mathbb{A}$ has *symmetric polymorphisms* of all arities. An operation $f \colon A^n \to A$is symmetric if it satisfies the identity $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all permutations $\pi \in \mathrm{Sym}(n)$.

Problem 5. Conclude that BLP solves CSP($\mathbb{A}$) if and only if Pol($\mathbb{A}$) contains symmetric operations of all arities.

Problem 6. Show that this is further equivalent to the existence of a minion homomorphism from the clone of convex linear functions on $\mathbb{Q}$, i.e. $\{\mathbf{x} \mapsto \sum_{i=1}^n p_i x_i \mid 0 \leq p_i \leq 1, \sum_{i=1}^n p_i = 1\}$ to Pol($\mathbb{A}$).

Problem 7. Discuss which of the tractable CSPs we discussed throughout the lecture can be solved by BLP or not (in particular: minimal tractable Boolean CSPs, bipartite graphs, linear equations over $\mathbb{Z}_p$, semilattices)?

Problem 7. Alternatively, we define the *AIP-relaxation* (affine integer programming relaxation) $AIP_{\mathbb{A}}(\mathbb{X})$ of a CSP instance of $\mathbb{X}$ as in Problem 1, by considering variables $\lambda_x(a)$ and $\lambda_C(\bar{a})$ *over the integers* $\mathbb{Z}$, such that $\sum_{a\in A}\lambda_x(a) = 1$, $\sum_{\bar{a}\in R^{\mathbb{A}}}\lambda_C(\bar{a}) = 1$ and the compatibility condition (1) holds. Since $AIP_{\mathbb{A}}(\mathbb{X})$ is a system of linear equations over $\mathbb{Z}$, it can be solved in polynomial time.

Discuss analogues of Problem 1-4 for AIP-relaxation.

Problem 8. Show that AIP solves CSP($\mathbb{A}$) if and only if Pol($\mathbb{A}$) contains *alternating operations* of all odd arities $2l+1$, i.e. operations $t(x_1,\dots,x_{2l+1})$, such that

- t is invariant under all permutations of variables that preserve the parity of indices
- $t(x_1,\dots,x_{2l-1},x,x) = t(x_1,\dots,x_{2l-1},y,y)$ for all $x_1,\dots,x_{2l-1},x,y \in A$.

For example $t(x_1,\dots,x_{2l+1}) = x_1 - x_2 + x_3 - \dots + x_{2l+1}$ is an alternating operations for any abelian group $+$.

Problem 9. Try to find a clone $\mathcal{C}$ such that AIP solves CSP($\mathbb{A}$) if and only if there is a minion homomorphism from $\mathcal{C}$ to Pol($\mathbb{A}$).

Problem 10. Which of the tractable CSPs we discussed throughout the lecture can be solved by AIP?