2^ω many maximal-closed subgroups of Sym(ω) via Henson digraphs

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New Pathways between Group Theory and Model Theory
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Closed subgroups of Sym(ω)

Sym(ω) is a topological group with the basis of clopen subgroups:

$\{f \in \text{Sym}(\omega) : f \restriction_A = id \restriction_A\}$ for all finite $A \subset \omega$.

A closed subgroup $\Sigma < \text{Sym}(\omega)$ is maximal-closed if there is no closed Σ' with $\Sigma < \Sigma' < \text{Sym}(\omega)$.

Question (Macpherson)

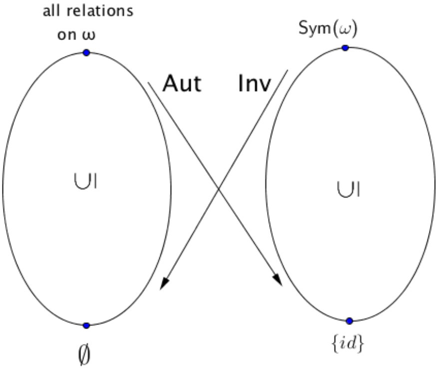
Are there 2^ω non-conjugate maximal-closed subgroups of Sym(ω)?

Conjugation describes permutations up to renaming of elements:

If $f = \kappa^{-1} g \kappa$ then $y = f(x) \leftrightarrow \kappa(y) = g(\kappa(x))$

Reducts of ω-categorical structures

Aut($\mathcal{A}$): automorphisms of relational structure $\mathcal{A} = (\omega, R_1, R_2, \ldots)$
Inv(F): relations preserved by $F \subseteq$ Sym(ω)



Aut(Inv(F)): Minimal closed group containing F

$\mathcal{A}$ is reduct of $\mathcal{A}'$ or $\mathcal{A} \leq_{f.o.} \mathcal{A}'$ if every relation in $\mathcal{A}$ is definable in $\mathcal{A}'$.

$\mathcal{A} \leq_{f.o.} \mathcal{A}' \to \text{Aut}(\mathcal{A}) \geq \text{Aut}(\mathcal{A}')$.

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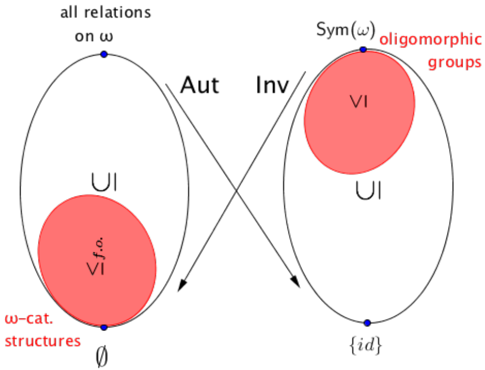

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$\mathcal{A} \leq_{f.o.} \mathcal{A}' \rightarrow \mathrm{Aut}(\mathcal{A}) \geq \mathrm{Aut}(\mathcal{A}')$.

For ω-categorical $\mathcal{A}'$ structures: $\mathcal{A} \leq_{f.o.} \mathcal{A}' \leftrightarrow \mathrm{Aut}(\mathcal{A}) \geq \mathrm{Aut}(\mathcal{A}')$.

Examples

Thomas '91

There are exactly 5 reducts of the Random graph.

For $n > 2$ there is a unique countable graph $(H_n, \bar{E})$ such that

- $(H_n, \bar{E})$ embeds all graphs not containing K_n,
- $(H_n, \bar{E})$ is homogenous, ω-categorical.

We call $(H_n, \bar{E})$ a Henson graph.

Thomas '91

Every Henson graph $(H_n, \bar{E})$ has only trivial reducts. So $\mathrm{Aut}(H_n, \bar{E})$ is a maximal-closed subgroup of $\mathrm{Sym}(H_n)$.

Henson digraphs

$\mathcal{T}$ ... set of finite tournaments.

There is a unique countable digraph $(D_\mathcal{T}, E)$ such that

- Finite substructures of $(D_\mathcal{T}, E)$ are exactly the digraphs that don't embed tournaments from $\mathcal{T}$,
- $(D_\mathcal{T}, E)$ is homogeneous, ω-categorical.

Let $\mathcal{T} \neq \emptyset$ and not contain the 2-tournament. Then $(D_\mathcal{T}, E)$ is called a Henson digraph.

There are 2^ω Henson digraphs.

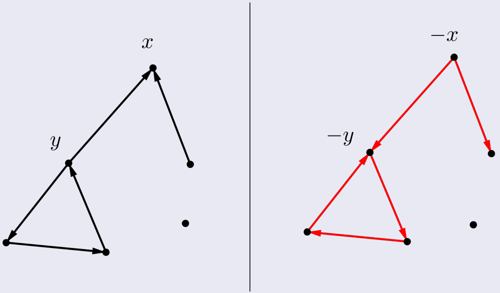
Question

What are the reducts for a given Henson digraph $(D_\mathcal{T}, E)$?

Possible reducts

Example

Let T be closed under switching the direction of all edges. Then there is a bijection $- : D_T \to D_T$ such that $E(x,y) \leftrightarrow E(-y,-x)$.

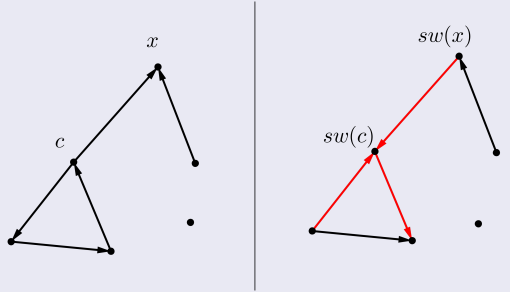


The closed group $\langle - \rangle$ generated by $\mathrm{Aut}(D_T, E) \cup \{-\}$ is a proper supergroup of $\mathrm{Aut}(D_T, E)$.

Possible reducts

Example

Assume there is a bijection $sw : D_T \to D_T$ that switches all edges adjacent to a vertex c, while preserving all other edges:



The existence of sw only depends on T.
If it exists $\langle sw \rangle$ is a closed supergroup of $\mathrm{Aut}(D_T, E)$.

The reducts of Henson digraphs

Let $\bar{E}(x,y) \leftrightarrow E(x,y) \vee E(y,x)$.

Theorem (Agarwal, MK '15)

Let (D_T, E) be a Henson digraph and $G \geq \mathrm{Aut}(D_T, E)$. Then

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- If $G < \mathrm{Aut}(D_T, \bar{E})$, then $G = \mathrm{Aut}(D_T, E)$, $\langle - \rangle$, $\langle sw \rangle$ or $\langle -, sw \rangle$

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- If $G < \mathrm{Aut}(D_T, \bar{E})$, then $G = \mathrm{Aut}(D_T, E)$, $\langle - \rangle$, $\langle sw \rangle$ or $\langle -, sw \rangle$
- One of the following holds
  - $(D_T, \bar{E})$ is the Random graph
  - $(D_T, \bar{E})$ is a Henson graph
  - $(D_T, \bar{E})$ has no proper reducts and $\mathrm{Aut}(D_T, \bar{E}) = \max\{\mathrm{Aut}(D_T, E), \langle - \rangle, \langle sw \rangle, \langle -, sw \rangle\}$

The reducts of Henson digraphs

$\mathrm{Sym}(D_T)$

$\langle sw_{\bar{E}}, -_{\bar{E}} \rangle$

$\langle sw_{\bar{E}} \rangle$ $\quad$ $\langle -_{\bar{E}} \rangle$

$\mathrm{Aut}(D_T, \bar{E})$

$\langle sw, - \rangle$

$\langle sw \rangle$ $\quad$ $\langle - \rangle$

$\mathrm{Aut}(D_T, E)$

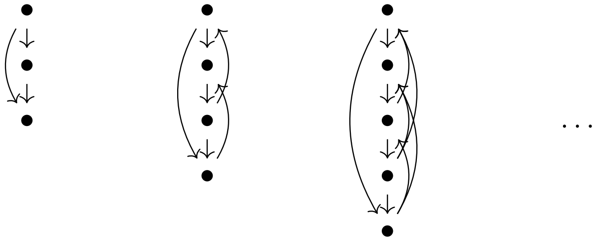
The lattice to the left shows all potential closed supergroups of $\mathrm{Aut}(D_T, E)$.

$\mathrm{Aut}(D_T, E)$ is maximal-closed if

- T not closed under $-$ and sw
- $(D_T, \bar{E})$ is not homogeneous

2^ω non-isomorphic Henson digraphs

Let I_n be the tournament we obtain by taking a linear order of size n and flipping all edges $(i, i+1)$ for $i < n$ and $(1, n)$.



The Henson digraphs (D_T, E) for $T \subseteq \{I_n : n > 6\}$ are pairwise non-isomorphic.

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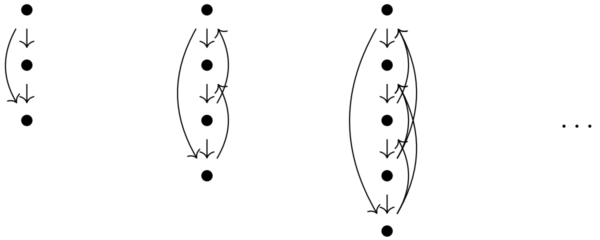

The Henson digraphs (D_T, E) for $T \subseteq \{I_n : n > 6\}$ are pairwise non-isomorphic.

Let X have a sink, but no source and be not embeddable in any I_n.

Let

$$T' = \{X' : |X'| = |X| + 1, X \subset X'\} \cup T.$$

Then $(D_{T'}, E)$ has only trivial reducts.

Reconstruction

Conclusion

There are 2^ω many Henson digraphs $(D_{T'}, E)$ such that $\mathrm{Aut}(D_{T'}, E)$ are maximal-closed and pairwise non-conjugate.

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Rubin '87

For two Henson digraphs (D_1, E) and (D_2, E) the following are equivalent:

- $\mathrm{Aut}(D_1, E)$ and $\mathrm{Aut}(D_2, E)$ are conjugate
- $\mathrm{Aut}(D_1, E) \cong_T \mathrm{Aut}(D_2, E)$
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There are 2^ω many Henson digraphs $(D_{T'}, E)$ such that $\mathrm{Aut}(D_{T'}, E)$ are maximal-closed and pairwise non-conjugate.

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Corollary

There are 2^ω many *non-isomorphic* maximal-closed subgroups of $\mathrm{Sym}(\omega)$.

Thank you!