An introduction to Ramsey theory

Michael Kompatscher

michael@logic.at

Theory and Logic group
TU Wien

Podzimní škola, 26.11.2016

What is Ramsey theory?

The philosophy of Ramsey theory:
In every *large enough* system there are regular subsystems.

What is Ramsey theory?

The philosophy of Ramsey theory:
In every *large enough* system there are regular subsystems.

Most basic example: **the pigeonhole principle**

k holes but $\geq k+1$ pigeons $\rightarrow$ One hole contains more than 1 pigeon.

What is Ramsey theory?

The philosophy of Ramsey theory:
In every *large enough* system there are regular subsystems.

Most basic example: **the pigeonhole principle**

k holes but $\geq 2k+1$ pigeons $\rightarrow$ One hole contains more than 2 pigeons.

The pigeonhole principle

More formally:
Let X be a finite set $|X| \geq kn + 1$ (the "pigeons").
Let $X = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ be a partition/coloring of A (the "pigeonholes").

Then $\exists i : |\mathcal{A}_i| \geq n + 1$.

The pigeonhole principle

More formally:
Let X be a finite set $|X| \geq kn + 1$ (the "pigeons").
Let $X = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ be a partition/coloring of A (the "pigeonholes").

Then $\exists i : |\mathcal{A}_i| \geq n + 1$.

An example

- X... inhabitants of Prague $|X| \geq 1.200.000$
- k... maximum number of hair on a human head $k = 250.000$
- $\mathcal{A}_i$... Praguer with i many hairs on their head

By pigeonhole principle there are at least 4 people in Prague with the same number of hair on their head!

A fact about scientific talks

Suppose you give a talk and more than 6 people are in the audience X.

A fact about scientific talks

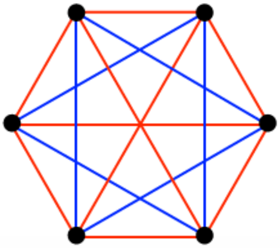
Suppose you give a talk and more than 6 people are in the audience X.

Claim: Then there are at least 3 people X that are mutual acquaintances or three guests will be mutual strangers.

A fact about scientific talks

Suppose you give a talk and more than 6 people are in the audience X.

Claim: Then there are at least 3 people X that are mutual acquaintances or three guests will be mutual strangers.

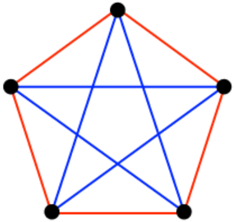


Color pairs of people $[X]^2$ red if they know each other; blue otherwise.

Proof: on board.

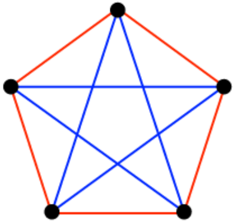
A fact about scientific talks

However, the statement is not true, if $|X| = 5$:



A fact about scientific talks

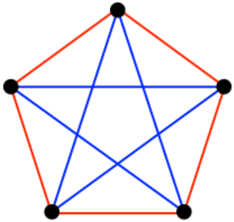
However, the statement is not true, if $|X| = 5$:



Generalisations:

A fact about scientific talks

However, the statement is not true, if $|X| = 5$:



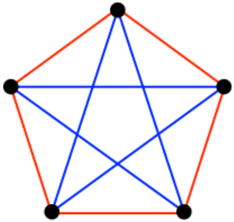
Generalisations:

- Is there a big enough X, such that we find n mutual acquaintances/strangers for $n = 4, 5, 6, \ldots$?

By Ramsey's theorem the answer is always **YES**.

A fact about scientific talks

However, the statement is not true, if $|X| = 5$:



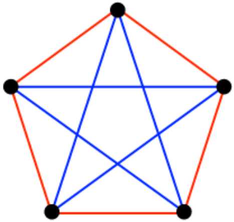
Generalisations:

- Is there a big enough X, such that we find n mutual acquaintances/strangers for $n = 4, 5, 6, \ldots$?
- What if we color the edges with more than 2 colors?

By Ramsey's theorem the answer is always **YES**.

A fact about scientific talks

However, the statement is not true, if $|X| = 5$:



Generalisations:

- Is there a big enough X, such that we find n mutual acquaintances/strangers for $n = 4, 5, 6, \ldots$?
- What if we color the edges with more than 2 colors?
- What if we color 3-sets, 4-sets, p-sets instead?

By Ramsey's theorem the answer is always **YES**.

Ramsey's theorem for pairs

We only show the statement for 2-colorings of pairs:

Ramsey's theorem for pairs

We only show the statement for 2-colorings of pairs:

Finite Ramsey theorem

$\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every partition $[N]^2 = \mathcal{A}_1 \cup \mathcal{A}_2$, there is an $i \in \{0,1\}$ and an n-subset $B \subseteq N$ with $[B]^2 \subseteq \mathcal{A}_i$.

Ramsey's theorem for pairs

We only show the statement for 2-colorings of pairs:

Finite Ramsey theorem

$\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every partition $[N]^2 = \mathcal{A}_1 \cup \mathcal{A}_2$, there is an $i \in \{0, 1\}$ and an n-subset $B \subseteq N$ with $[B]^2 \subseteq \mathcal{A}_i$.

Proof: $N = 2^{2n}$... proof on board

Ramsey's theorem for pairs

We only show the statement for 2-colorings of pairs:

Finite Ramsey theorem

$\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every partition $[N]^2 = \mathcal{A}_1 \cup \mathcal{A}_2$, there is an $i \in \{0,1\}$ and an n-subset $B \subseteq N$ with $[B]^2 \subseteq \mathcal{A}_i$.

Proof: $N = 2^{2n}$... proof on board

Note that N is not optimal; finding the optimal *Ramsey number* for n is a difficult task and unknown for $n = 5$.

Ramsey's theorem for pairs

We only show the statement for 2-colorings of pairs:

Finite Ramsey theorem

$\forall n \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every partition $[N]^2 = \mathcal{A}_1 \cup \mathcal{A}_2$, there is an $i \in \{0, 1\}$ and an n-subset $B \subseteq N$ with $[B]^2 \subseteq \mathcal{A}_i$.

Proof: $N = 2^{2n}$... proof on board

Note that N is not optimal; finding the optimal *Ramsey number* for n is a difficult task and unknown for $n = 5$.

Infinite Ramsey theorem

For every partition of pairs of the natural numbers $[\mathbb{N}]^2 = \mathcal{A}_1 \cup \mathcal{A}_2$, there is an $i \in \{0, 1\}$ and an infinite subset $X \subseteq \mathbb{N}$ with $[X]^2 \subseteq \mathcal{A}_i$.

Applications of Ramsey's theorem

Why did Ramsey prove his theorem?

Applications of Ramsey's theorem

Why did Ramsey prove his theorem?

Ramsey was interested in Hilbert's *Entscheidungsproblem*: Is there an algorithm, telling us if a given first-order formula is provable?

Applications of Ramsey's theorem

Why did Ramsey prove his theorem?

Ramsey was interested in Hilbert's *Entscheidungsproblem*: Is there an algorithm, telling us if a given first-order formula is provable?

He used the theorem to show that there is such an algorithm for formulas of the form

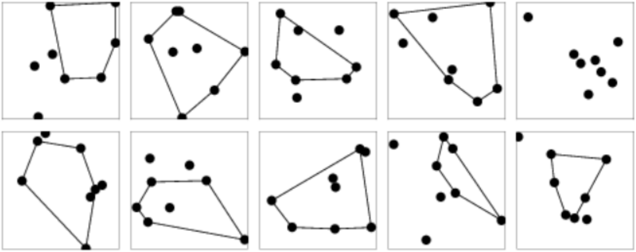
$$\exists x_1, \dots, x_n \forall y_1, \dots, y_k \phi(x_1, \dots, x_n, y_1, \dots, y_k).$$

(In general Entscheidungsproblem is undecidable!)

The happy ending theorem

The happy ending theorem (Erdös, Szekeres '35)

For any positive integer n there is an N, such that every set of N points $X \subset \mathbb{R}^2$ in general position has n points that form the vertices of a convex polygon.



fail for $n = 5$, $N = 8$

Results before Ramsey

Schur's theorem (1919)

For every k and any partition

$$\mathbb{N} = A_1 \cup \cdots \cup A_k,$$

one of the partition classes A_i contain a set $\{x, y, z\}$ with $x + y = z$.

Results before Ramsey

Schur's theorem (1919)

For every k and any partition

$$\mathbb{N} = A_1 \cup \cdots \cup A_k,$$

one of the partition classes A_i contain a set $\{x, y, z\}$ with $x + y = z$.

Schur used this result to make statements about Fermat's last theorem for finite fields:

Results before Ramsey

Schur's theorem (1919)

For every k and any partition

$$\mathbb{N} = A_1 \cup \cdots \cup A_k,$$

one of the partition classes A_i contain a set $\{x, y, z\}$ with $x + y = z$.

Schur used this result to make statements about Fermat's last theorem for finite fields:

There is no solution of

$$x^n + y^n = z^n \mod p$$

for every p large enough.

Results before Ramsey

Van der Waerden's theorem (1927)

For all $k, n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that for every partition $N = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$ there is an index i_0, such $\mathcal{A}_{i_0}$ contains an *arithmetic progression* of length n, i.e. a set of the form

$$\{a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (n-1)d\},$$

where $a_0, d \in \mathbb{N}$.

The best known upper bound for N are very bad...

$$N(n,k) \leq 2^{2^{n^{2^{2^{k+9}}}}}$$

... and there is a prize of 1000$ for showing $W(2,k) < 2^{k^2}$.

Structural Ramsey theory

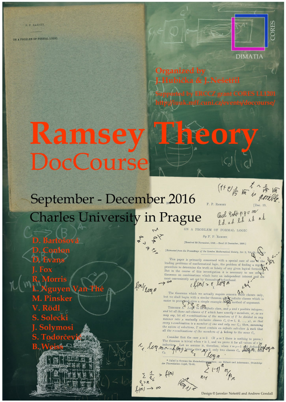
What if, instead of subsets, we color

- edges of (non-complete) graphs?
- subgraphs of graphs?
- subspaces of vector spaces?
- subalgebras of Boolean algebras?
- ...

Results for all of the above are known and discussed in *Structural Ramsey theory*.

Ramsey DocCourse 2016

http://iuuk.mff.cuni.cz/events/doccourse/



Mike Pawliuk's Blog provides good notes of the program:
https://boolesrings.org/mpawliuk/

Thank you!