# Minimal Mal'cev clones

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# Mal'cev CSPs

$m: A^3 \to A$ is a Mal'cev operation if

$$m(y\,x\,x) \approx m(x\,x\,y) = y$$

Examples:

- heap $\qquad m(xyz) = x y^{-1} z$ over a group $(G, \cdot)$
- minority$_{\Pi_1}$ $\qquad m(xyz) = \begin{cases} \min(xyz) & \text{if } |\{x,y,z\}| \leq 2 \\ x & \text{else} \end{cases}$

**Theorem (Bulatov, Dalmau '07)**

If $\exists\, m \in \mathrm{Pol}(\mathbb{A})$ Mal'cev

$\Rightarrow \mathrm{CSP}(\mathbb{A}) \in P$

⚠ This algorithm depends on $m(xyz)$

↳ not uniform

# The quest for THE algorithm

several uniform (minion-relaxation) algorithms fail for Mal'cev / group CSPs $\mathrm{CSP}(\mathbb{G})$

$$\mathrm{Pol}(\mathbb{G}) = \mathrm{Clo}(G, xy^{-1}z)$$

$$\langle \mathbb{G} \rangle_{pp} = \mathrm{Inv}(xy^{-1}z) = \{aH \mid H \leq G^n, a \in G^n\}$$

$\mathrm{CSP}(\mathbb{D}_4)$ not solved by $\mathrm{Sing}(\mathrm{BLP}+\mathrm{AIP})$ (Zhuk '25)

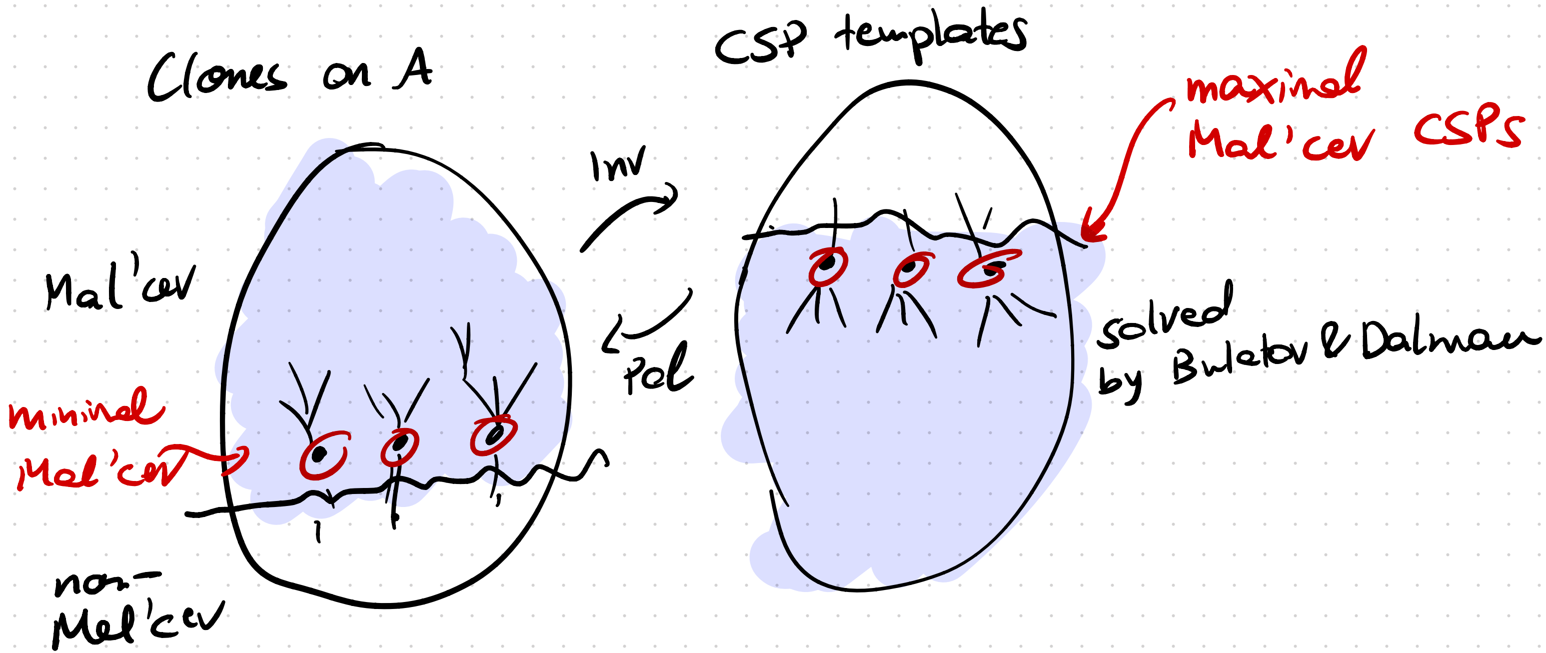
Task: Find a uniform algorithm solving
- Mal'cev
- solvable Mal'cev
- group
- 2-group
- ⋮

CSPs

# Minimal Mal'cev clones

A clone $\mathcal{C}$ is a minimal Mal'cev clone if

$$\mathcal{C} = Clo(m) \quad \forall \text{ Mal'cev } m \in \mathcal{C}$$



# Examples

- Abelian heaps $(A, x-y+z)$
  ↪ $x-y+z$ is <u>only</u> Mal'cev operation
  ↪ CSP($\mathbb{A}$) is solved by AIP
- on $|A|=2$ : $\mathbb{Z}_2$
- on $|A|=3$ : (by Bulatov '05)
  $\mathbb{Z}_3$, $\min_{\pi_1}$, $\min_{const}$ ↪ solved by Sing(AIP)

All examples are minimal Taylor $\Leftrightarrow$ maximally tractable

↪ Does this hold in general? Zeb: Yes!

## Basic properties

$\mathcal{C} = \mathrm{Clo}(m)$ minimal Mal'cev clone on $A$

- $B \leq A$ invariant under some $d \in \mathcal{C}$, $d\restriction_B$ Mal'cev

  $\Rightarrow$ $\mathcal{C}\restriction_B$ is minimal Mal'cev clone

Proof: $d_1(xyz) := m(d(xyz), d(xyy), x)$

$d_2(xyz) := m(z, d_1(yyz), d_1(xyz))$

$\Rightarrow d_2$ is Mal'cev on $A$, $d_2\restriction_B = d\restriction_B$ $\square$

- If $\mathcal{D} = \xi(\mathcal{C})$ for clone homomorphism.

  $\Rightarrow$ $\mathcal{D}$ minimal Mal'cev clone.

# minimal Taylor

**Theorem** (Brady '25, ~~unpublished~~)

$\mathcal{C}$ is minimal Mal'cev $\Rightarrow$ $\mathcal{C}$ is minimal Taylor.

**Proof idea:**

Let $\mathcal{D} \subseteq \mathcal{C}$ be minimal Taylor.

If $\mathcal{D}$ not Mal'cev $\Rightarrow$ 1) $\exists\, a,b$ $\quad x \wedge y \in \mathcal{D}$ semilattice on $\{a,b\}$ or

or 2) $\exists\, a,b$ $\quad maj(x,y,z) \in \mathcal{D}$ majority on $\langle a,b \rangle / \theta$

2) If e.g. $\langle a,b \rangle = \{a,b\}$

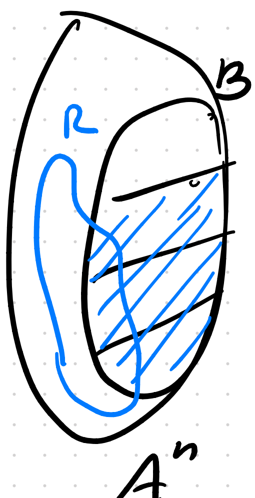
$m'(xyz) = m(x, maj(x,y,z), z)$ is Mal'cev on $\{a,b\}$

$\Rightarrow \mathcal{C} \restriction \{a,b\}$ minimal Mal'cev, but has majority ↯ $\quad \ldots \square$

# Near invariant relations

How to find minimal Mal'cev reducts of some $\underline{A}=(A,m)$?

<u>Def</u>. $R \subseteq A^n$ is near-invariant under $\underline{A}$ if
$\forall \underline{B} \leq \underline{A}^n \ \forall \alpha : \underline{B}/\alpha$ Abelian $\Rightarrow (B \cap R)/\alpha \leq \underline{B}/\alpha$




If $\underline{C}$ is Mal'cev reduct of $\underline{A}$
$\Rightarrow$ all $R \leq \underline{C}^n$ are near-invariant under $\underline{A}$

[since $\underline{B}/\alpha = (B/\alpha, x-y+z)$ is minimal Mal'cev]
affine

# Minimal Mal'cev reducts of groups

**Theorem (Feder '05)**

1) $\underline{G} = (G, xy^{-1}z)$

$\mathbb{G}^+ := \{ R \subseteq G^n \mid R \text{ near-invariant under } xy^{-1}z \}$

has Mal'cev polymorphism $m(xyz) = xy^{-1}z \cdot \underbrace{c(xy^{-1}, zy^{-1})}_{\text{c commutator expression}}$

$\Rightarrow$ $\underline{G}$ has a unique minimal Mal'cev reduct

2) If $|G| = 2^n$ (2-group), then

$\underline{G}$ is minimal Mal'cev itself!

# A minion description of 2-groups

$\underline{G} = (G, \cdot)$ finite group

terms = words

Ex.: $t(x_1,x_2,x_3) = x_2 x_1 x_2 x_1 x_3 x_3 x_3$

$f_t(s) :=$ number of subwords $s$ in $t$

Ex.: $f_t(x_2)=2 \quad f_t(x_2 x_1) = 3 \quad f_t(x_3 x_3) = 3$

$f_t$ satisfies some obvious identities:

$$f_t(x_1)\cdot f_t(x_2) = f_t(x_1 x_2) + f_t(x_2 x_1)$$

$$f_t(x_1)^2 = 2 f_t(x_1 x_1) + f_t(x_1)$$

# A minion description of 2-groups

**Theorem** (Thérien '83) $\underline{G}$ $n$-nilpotent.

if $f_t(s) = f_{t'}(s)$ $\forall |s| \leq n$ $\Rightarrow$ $G \models t \approx t'$

**similarly** (MK, '26+) $\underline{G}$ 2-group $\Rightarrow \exists n$

if $f_t(s) = f_{t'}(s)$ mod 2 $\forall |s| \leq n$ $\Rightarrow$ $G \models t \approx t'$

**Idea:** Consider the minion $\mathcal{M}_n$ of all functions $f : X^{\leq n} \to \mathbb{Z}_2$ satisfying the obvious identities.

$$\mathrm{Clo}(\underline{G}) \to \mathcal{M}_n$$

# A minion description of 2-groups

**Example** $n = 2$

$(f_1, f_2) \in \mathcal{M}_2^{(X)} \Leftrightarrow f_1 : X \to \mathbb{Z}_2 \quad f_2 : X^2 \to \mathbb{Z}_2$

obv. identities $\begin{cases} f_2(x,y) + f_2(y,x) = f_1(x) \cdot f_1(y) & \forall x \neq y \\ \cancel{2 f_2(xx) = f_1(x)^2 - f_1(x)} \end{cases}$

idempotence $\quad \sum_x f_1(x) = 1 \quad \sum_x f_2(xx) = 0$

$\langle x y^{-1} z \rangle \quad \sum_x f_2(x\, y) = 0$

$= D_4$ minion

$n > 3$ …much uglier e.g. $f_2(x\, y) \cdot f_2(x\, z) = f_3(x\, y\, z) + f_3(x\, z\, y) + f_4(x\, y\, x\, z)$
$+ f_4(x\, z\, x\, y) + 2 f_4(x\, x\, y\, z) + 2 f_4(x\, x\, z\, y)$

**Task:** Show $\mathcal{M}_n$ solved by some level of $\mathbb{Z}_2$ (AIP)

Thank you!