Finitely tractable PCSPs

Michael Kompatscher
Charles University Prague

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The Constraint Satisfaction Problem: Complexity and Approximability

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Promise constraint satisfaction problems (PCSPs)

$\mathbb{A} = (A, R_1^{\mathbb{A}}, \ldots, R_n^{\mathbb{A}})$
$\mathbb{B} = (B, R_1^{\mathbb{B}}, \ldots, R_n^{\mathbb{B}})$
finite, with $\mathbb{A} \to \mathbb{B}$ homomorphism

PCSP($\mathbb{A}, \mathbb{B}$) **(decision version)**

INPUT: $\mathbb{X}$

OUTPUT: **Yes** if $\mathbb{X} \to \mathbb{A}$

No if $\mathbb{X} \not\to \mathbb{B}$

Example

- PCSP($\mathbb{K}_3, \mathbb{K}_5$): Is $\mathbb{X}$ 3-colorable, or not even 5-colorable?
- $\mathbb{A} = (\{0, 1\}$,1in3), $\mathbb{B} = (\{0, 1\}$,NAE):
  Is a list of triples $(x_1, x_3, x_5), (x_2, x_1, x_4), \ldots$
  1-in-3 satisfiable or not even NAE-satisfiable?
- PCSP($\mathbb{A}, \mathbb{A}$) = CSP($\mathbb{A}$)

Question: When does PCSP($\mathbb{A}, \mathbb{B}$) reduce to a finite CSP?

Sandwiches

If $\mathbb{C}$ is **sandwiched** between $\mathbb{A}$ and $\mathbb{B}$:

$$\mathbb{A} \to \mathbb{C} \to \mathbb{B},$$

then PCSP($\mathbb{A}, \mathbb{B}$) trivially reduces to CSP($\mathbb{C}$).

If $\exists\mathbb{C}$ **finite 'cheese'**, such that

- $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$
- CSP($\mathbb{C}$) $\in$ P,

then PCSP($\mathbb{A}, \mathbb{B}$) is called **finitely tractable**.

Examples

- CSP($\mathbb{A}$) $\in$ P or CSP($\mathbb{B}$) $\in$ P
  $\Rightarrow$ PCSP($\mathbb{A}, \mathbb{B}$) finitely tractable
- $(\{0,1\}, \text{1in4}) \to (\{0,1\}, \{\bar{x} \mid \sum_{i=1}^{4} x_i = 1\}) \to (\{0,1\}, \text{NAE}_4)$
  is a 'proper' sandwich witnessing finite tractability
- PCSP($\mathbb{K}_3, \mathbb{K}_5$) is not finitely tractable

Sandwiches

If $\mathbb{C}$ is **sandwiched** between $\mathbb{A}$ and $\mathbb{B}$:

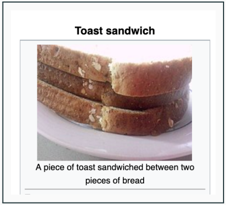
$$\mathbb{A} \to \mathbb{C} \to \mathbb{B},$$

then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ trivially reduces to $\mathrm{CSP}(\mathbb{C})$.

If $\exists \mathbb{C}$ **finite 'cheese'**, such that

- $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$
- $\mathrm{CSP}(\mathbb{C}) \in \mathrm{P}$,

then $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ is called **finitely tractable**.



Examples

- $\mathrm{CSP}(\mathbb{A}) \in \mathrm{P}$ or $\mathrm{CSP}(\mathbb{B}) \in \mathrm{P}$
  $\Rightarrow \mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ finitely tractable
- $(\{0,1\}, \text{1in4}) \to (\{0,1\}, \{\bar{x} \mid \sum_{i=1}^{4} x_i = 1\}) \to (\{0,1\}, \text{NAE}_4)$
  is a 'proper' sandwich witnessing finite tractability
- $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$ is not finitely tractable

Questions

Task: Characterize finitely tractable PCSP($\mathbb{A}, \mathbb{B}$)

Functional approach

- finite tractability is preserved under gadget reductions [AB21]

$\Rightarrow$ determined by polymorphism minion
$\text{Pol}(\mathbb{A}, \mathbb{B}) = \{f : \mathbb{A}^n \to \mathbb{B} \mid n \in \mathbb{N}\}$

- Which minor identities characterize finite tractability?

Structural approach

- For PCSP($\mathbb{A}, \mathbb{B}$), can we bound the minimal size of the tractable cheese $\mathbb{C}$? (Mayr)
- necessary conditions on $(R^{\mathbb{A}}, R^{\mathbb{B}})$?

Special case: Boolean PCSPs $|A| = |B| = 2$

Functional approach

Necessary minor identities

For $\mathbb{A} \to^g \mathbb{C} \to^h \mathbb{B}$:
$\mathrm{Pol}(\mathbb{C}) \to \mathrm{Pol}(\mathbb{A}, \mathbb{B})$, $t \mapsto h \circ t \circ (g, \ldots, g)$ is minion homomorphism.

$\Rightarrow$ finitely tractable $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ has

- Siggers polymorphisms $s(xyxzyz) \approx s(yxzxzy)$
- cyclic polymorphisms $c(x_1, \ldots, x_p) \approx c(x_2, \ldots, x_p, x_1)$, $\forall p > |C|$
- 'doubly cyclic' polymorphisms for $p > |C|$,
- $\cdots$

Examples

- $\mathbb{A} = (\{0,1\}, \text{1in3})$, $\mathbb{B} = (\{0,1\}, \text{NAE})$; $\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \in \mathrm{P}$
  no doubly cyclic polymorphism $\Rightarrow$ not finitely tractable (Barto 19)
- $\mathbb{A} = (\{0,1\}, \text{1in3}) = \mathbf{LO}_2^3$,
  $\mathbb{B} = (\{0,1,2\}, \{\overleftrightarrow{001}, \overleftrightarrow{002}, \overleftrightarrow{112}, \overleftrightarrow{012}\}) = \mathbf{LO}_3^3$,
  $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ not finitely tractable
  (no cyclic polymorphisms for $p = 4k + 3$).

Example

Asimi & Barto classified all tractable Boolean symmetric PCSPs allowing $(\neq, \neq)$ up to finite tractability:

Asimi, Barto '21

▶ **Theorem 3.** *The PCSP over any of the following templates is not finitely tractable.*

(1) $(r\text{-}in\text{-}s, \leq(2r-1)\text{-}in\text{-}s), (\neq, \neq)$ *where* $1 < r < s/2$,
$(r\text{-}in\text{-}s, \geq(2r-s+1)\text{-}in\text{-}s), (\neq, \neq)$ *where* $s/2 < r < s-1$

(2) $(\leq r\text{-}in\text{-}s, \leq(2r-1)\text{-}in\text{-}s), (\neq, \neq)$ *where s is even,* $1 < r = s/2$
$(\geq r\text{-}in\text{-}s, \geq(2r-s+1)\text{-}in\text{-}s), (\neq, \neq)$ *where s is even,* $1 < r = s/2$

(3) $(r\text{-}in\text{-}s, \leq(2r-1)\text{-}in\text{-}s), (\neq, \neq)$ *where s is even,* $1 < r = s/2$, *and r is even*
$(r\text{-}in\text{-}s, \geq(2r-s+1)\text{-}in\text{-}s), (\neq, \neq)$ *where s is even,* $1 < r = s/2$, *and r is even*

(4) $(r\text{-}in\text{-}s, not\text{-}all\text{-}equal\text{-}s)$ *where* $s > r$, $s > 2$, *and r is even or s is odd*

Otherwise: affine cheese $\mathbb{C}$ over $\mathbb{Z}_2$, e.g.

$(\{0,1\}, \text{1in4}) \to (\{0,1\}, \{\bar{x} \mid \sum_{i=1}^{4} x_i = 1 \mod 2\}) \to (\{0,1\}, \text{NAE}_4)$

Question: What about non-symmetric templates?

Bounded width cheese

Example: $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$
$A = B = C = \{0, 1\}$;
$R^{\mathbb{C}} = (x_1 = 0 \vee x_2 = 0) \wedge (x_3 = 1 \vee x_4 = 1)$
$R^{\mathbb{A}} = R^{\mathbb{C}} \setminus \{(0011)\}$
$R^{\mathbb{B}} = \mathrm{NAE}_4$

- CSP($\mathbb{C}$) has bounded width
- no alternating polymorphisms $\Rightarrow$ no affine cheese $\mathbb{C}'$

Bounded width cheese

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Theorem [MK '21]

- For $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$ with $|A| = |B| = 2$,
- and CSP($\mathbb{C}$) bounded width,

$\Rightarrow$ Pol($\mathbb{A}, \mathbb{B}$) has symmetric terms of all odd arities.

Bounded width cheese

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- and CSP($\mathbb{C}$) bounded width,

$\Rightarrow$ Pol($\mathbb{A}, \mathbb{B}$) has symmetric terms of all odd arities.

Remarks

- Not true for Pol($\mathbb{C}$) itself.
- **Corollary:** PCSP($\mathbb{A}, \mathbb{B}$) solved by BLP+AIP

Proof idea

Proof idea: study local behaviour of $\mathrm{Pol}(\mathbb{C})$ on $\{0,1\} \subseteq C$ (using [Brady '19])

there are $c, d \in C$, and terms s, m:

$$s(x_1; x_2, \ldots, x_n) = \begin{cases} x_1 \text{ if } x_1 = \ldots = x_n \\ c \text{ if } x_1 = 0, \{x_1, \ldots, x_n\} = \{0,1\} \\ d \text{ if } x_1 = 1, \{x_1, \ldots, x_n\} = \{0,1\} \end{cases}$$

$$m(x_1, \ldots, x_n) = \mathrm{maj}(x_1, \ldots, x_n) \text{ if } x_1, \ldots, x_n \subseteq \{c, d\}$$

then $m(s(x_1; x_2, \ldots, x_n), \ldots, s(x_n; x_2, \ldots, x_n, x_1))|_{\{0,1\}}$ is symmetric.

$\square$

Question: Is there actually an example with $|C| > 2$?

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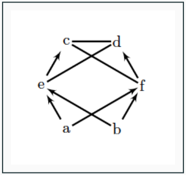

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□

Question: Is there actually an example with $|C| > 2$?

Structural approach

Big cheeses

Example (Kazda, Mayr, Zhuk '21)
$\mathbb{A} = (\{0,1\}, \{\pi_i^p \colon \{0,1\}^p \to \{0,1\} \text{ projection }\})$,
$\mathbb{B} = (\{0,1\}, \{f \colon \{0,1\}^p \to \{0,1\} \mid f \text{ not cyclic }\})$

- $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ has no p-cyclic polymorphims

$\Rightarrow$ no cheese of size $< p$

- but $\exists \mathbb{C} = (\mathbb{Z}_p; R^{\mathbb{C}})$ affine, with $\mathbb{A} \to \mathbb{C} \to \mathbb{B}$

$\Rightarrow$ For finitely tractable Boolean PCSPs $|C|$ cannot be bounded!

Question
Is there a bound on $|C|$, depending on $|A|, |B|$ **and arity**$(\mathbb{A})$?

Question (Barto)
Are there finitely tractable **symmetric** $\mathbb{A}$, $\mathbb{B}$ such that $|C| > |A|, |B|$?

A new loop lemma

Theorem [Zhuk, (MK) '22]
Let $R \subseteq C^{2k+1}$ for $k \geq 1$, $C = \mathcal{O} \sqcup \mathcal{I}$

- R symmetric
- $R \neq \emptyset$
- R invariant under WNU

$\Rightarrow R \cap \mathcal{O}^{2k+1} \neq \emptyset$ or $R \cap \mathcal{I}^{2k+1} \neq \emptyset$.

Corollary
If PCSP($\mathbb{A}, \mathbb{B}$) is a *symmetric* PCSP,

- $|B| = 2$
- $\exists R$ odd arity; $(0, \ldots, 0), (1, \ldots, 1) \notin R^{\mathbb{B}}$

$\Rightarrow$ PCSP($\mathbb{A}, \mathbb{B}$) is not finitely tractable.

Thank you!