CC-circuits and the expressive power of nilpotent algebras

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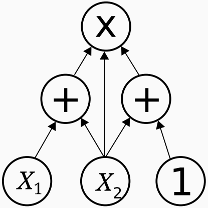
Birkhoff seminar, CU Boulder

Circuits in Universal Algebra: Why?

Circuits

Definition

A circuit is finite directed acyclic graph, where every vertex ('gate') is labelled by an operation of arity corresponding to its in-degree ('fan-in').



- natural model of computation
- usually studied for Boolean values
- Circuit over an algebra $\mathbf{A} = (A, f_1, \ldots, f_n)$: labelled by basic operations f_i

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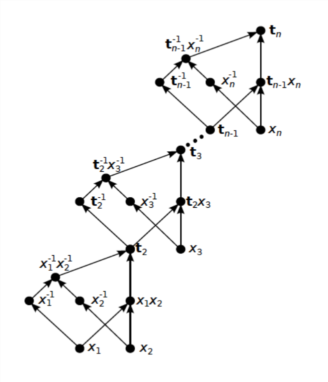
Example

In $(A_4, \cdot, ^{-1})$, the operations $t_n(x_1, \ldots, x_n) = [\cdots[[x_1, x_2], x_3], \ldots, x_n]$ can be represented by circuits linear in n, corresponds to terms exponential in n.



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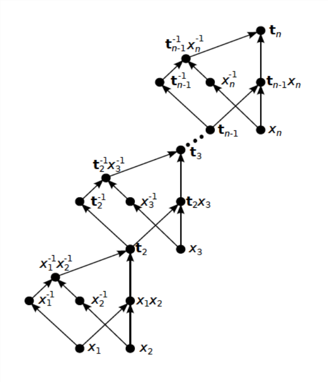
Encoding by circuits is

- more compact than encoding by terms
- stable under term equivalence

$\rightsquigarrow$ use in algorithmic problems.



Outline of this talk:

1. Circuit complexity and CC-circuits
2. Circuits over $\mathbf{A} \leftrightarrow$ CC-circuits
   for finite nilpotent $\mathbf{A}$ from CM varieties
3. Consequences in circuit complexity
4. Consequences for solving equations and checking identities
   in nilpotent algebras.

1) CC-circuits

Circuit complexity

Boolean circuits can be used to measure the complexity of $L \subseteq \{0,1\}^*$.

Basic idea

We say a family $(C_n)_{n \in \mathbb{N}}$ computes $L \subseteq \{0,1\}^*$ if $C_n(x_1, \dots, x_n) = 1 \leftrightarrow (x_1, \dots, x_n) \in L \cap \{0,1\}^n$. The complexity is measured by the size/depth of C_n.

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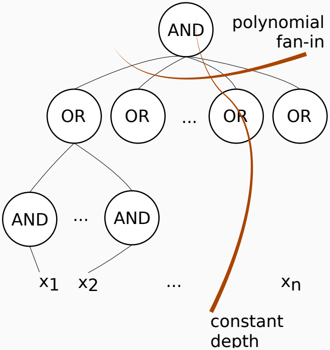
- AC^0: polynomial size, constant depth, but arbitrary fan-in



A result about AC^0-circuits

Theorem (Furst, Saxe, Sipser '84)

The parity language $\{x \in \{0,1\}^* : \sum_{i=1}^{n} x_i = 0 \mod 2\}$ is not in AC^0.

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Question:

- Are vice-versa counting gates bad at logic?
- What are circuits with 'counting gates'?

CC-circuits

A $CC[m]$-**circuit** is a (Boolean) circuit, whose gates are MOD_m-gates:

$$\text{MOD}_m(x_1, \ldots, x_n) = \begin{cases} 1 \text{ if } \sum_i x_i \equiv 0 \mod m \\ 0 \text{ else.} \end{cases}$$

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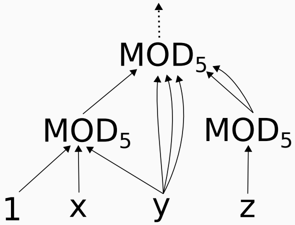

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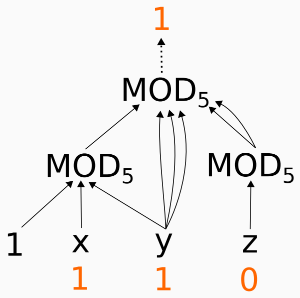

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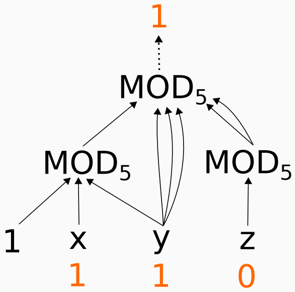

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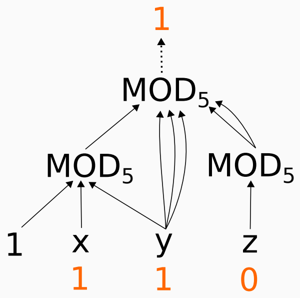

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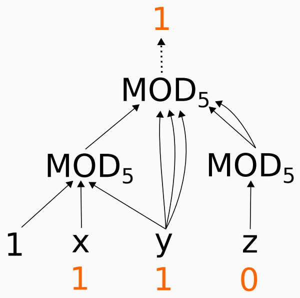

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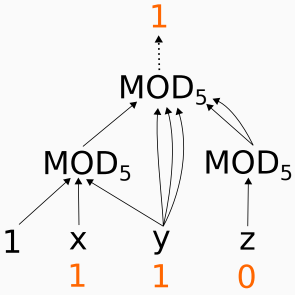

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- $CC^0 = \bigcup_m CC^0[m]$

A conjecture about CC-circuits

Conjecture (McKenzie*, Péladeau, Therién...)

$\forall m, d$: $CC[m]$-circuits of depth d need size $\Omega(e^n)$ to compute $\text{AND}(x_1, \ldots, x_n)$.

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- best known lower bounds in general are super-linear (CGPT '06)

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An operation f is called (0-)*absorbing* if
$f(0, x_2, \ldots, x_n) \approx f(x_1, 0, x_2, \ldots, x_n) \approx \cdots \approx f(x_1, \ldots, x_{n-1}, 0) \approx 0.$

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Lemma (MK '19)

$CC^+[m]$-circuit		$CC[m]$-circuit
non-trivial absorbing, depth d	$\rightarrow$	computing AND, depth d
non-trivial absorbing, depth $d+1$	$\leftarrow$	computing AND, depth d

$\rightarrow$... linear time computation

2) Nilpotent algebras

The structure of nilpotent algebras

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Also true for polynomial operations of $\mathbf{A}$

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- **B** contains the group $(B, +) = \mathbb{Z}_m^{d+1}$
- for every $CC[m]^+$-circuit C of depth d,
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Question

What about the opposite direction?

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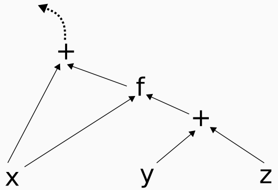

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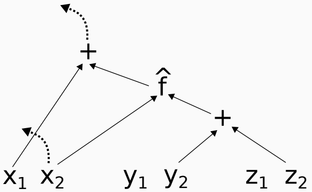

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$\mathbf{A} = (\mathbb{Z}_3 \times \mathbb{Z}_3, +, f(x,y))$ with

$$f((x_1,x_2),(y_1,y_2)) = (\hat{f}(x_2,y_2),0) = \begin{cases}(1,0) \text{ if } x_2 = y_2 = 1 \\ (0,0) \text{ else}\end{cases}$$

$\mathbf{A}$ is 2-nilpotent. Polynomial e.g.:
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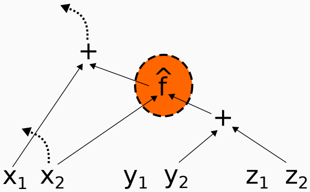

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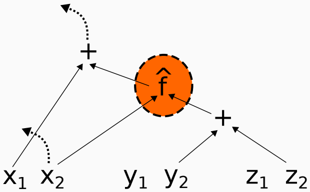

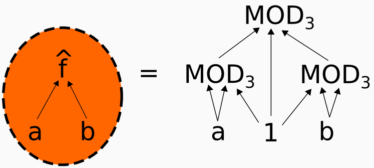

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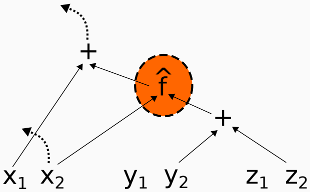

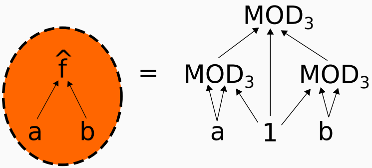

⇒ similarly all polynomials of **A** can be rewritten in polynomial time to $CC[3]^+$-circuits of depth 3

Coordinatisation of nilpotent algebras

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Theorem (Aichinger '18)

Let **A** be nilpotent, $|A| = p_1^{i_1} \cdot p_2^{i_2} \cdots p_m^{i_m}$. Then there are operations $+$,0,$-$ such that

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Remark

The degree of nilpotency might increase (but $\leq \log_2(|A|)$).
E.g. $(\mathbb{Z}_4, +)$ Abelian, but $(\mathbb{Z}_4, +, +_V)$ is 2-nilpotent.

Main result

A... finite nilpotent algebra (from CM variety)

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- If m is not prime power, then $C(\mathbf{A})$ is linear in $\log_2 |A|$.

3) Consequences on CC-circuits

Conjecture (*) in nilpotent algebras

An operation $f : A^n \to A$ is called 0-**absorbing** iff
$f(0, x_2, \dots, x_n) \approx f(x_1, 0, x_2, \dots, x_n) \approx \dots \approx f(x_1, \dots, x_{n-1}, 0) \approx 0$.

CC-circuits	in nilpotent algebra **A**
Conjecture (*) Bounded depth $CC[m]$-circuits need size $\Omega(e^n)$ to compute AND. **Theorem (BST '90)** Bounded depth $CC[p^k]$-circuits cannot compute AND of arity $\geq C(d)$ **Theorem (BST '90)** Conjecture (*) is true for $m = pq$ and depth 2	

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Theorem (BST '90) Conjecture (*) is true for $m = pq$ and depth 2	**(Idziak, Kawałek, Krzaczkowski '18)** (**) is true for certain 2-nilpotent **A** with $\lvert A\rvert = p^k q^l$

Remark

There exists another algebraic characterization of CC^0 by NUDFA (non-uniform deterministic finite automata) over monoids.

Theorem (Barrington, Straubing, Therien '90)

$L \in$ complexity class	$\leftrightarrow$	L accepted by a NUDFA over M
AC^0	$\leftrightarrow$	M aperiodic monoid
CC^0	$\leftrightarrow$	M solvable group
ACC^0	$\leftrightarrow$	M solvable monoid
NC^1	$\leftrightarrow$	M non-solvable group

4) Consequences on CSAT and CEQV

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$\mathbf{A} = (A, f_1, \ldots, f_n)$... finite algebra

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Circuit Equivalence Problem CEQV($\mathbf{A}$)

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QUESTION: Does $\mathbf{A} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$?

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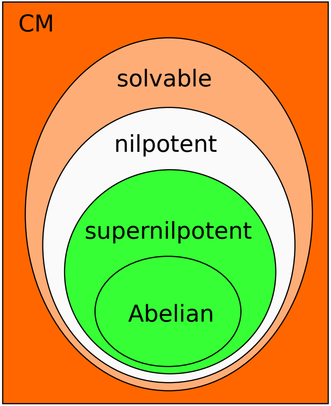
Question

What is the complexity for nilpotent **A** from CM varieties?

In congruence modular varieties

A... from congruence modular variety:



- **A** Abelian $\leftrightarrow$ module. CEQV(**A**) $\in$ P
- **A** k-supernilpotent. CEQV(**A**) $\in$ P:
  (Aichinger, Mudrinski '10)
- **A nilpotent, not supernilpotent...?**
- **A** solvable, non-nilpotent:
  $\exists\theta$: CEQV($\mathbf{A}/\theta$) $\in$ coNP-c
  (Idziak, Krzaczkowski '18)
- **A** non-solvable: CEQV(**A**) $\in$ coNP-c
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For CSAT the picture is similar (modulo products with DL algebras).

Circuit equivalence

Observation 1 (MK ’19)

Assume Conjecture (**) holds for **A** nilpotent.
Then CEQV(**A**) and CSAT(**A**) can be solved in quasipolynomial time.

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Note that for $|A| = p^j$: $k \leq const$
$\Rightarrow$ polynomial time algorithm for prime powers / supernilpotent.
(Aichinger, Mudrinski '10)

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Conclusion

Complexity of CEQV($\mathbf{A}$), CSAT($\mathbf{A}$) for nilpotent $\mathbf{A}$ is correlated to the expressive power of CC-circuits.

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(This is all we know, despite bold claims made at BLAST'19)

Thank you!