A complexity dichotomy for Poset-SAT

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Algebra Seminar, TU Wien, 08/04/2016

Outline

1. Poset-SAT problems
2. Poset-SAT as CSP over the random partial order
3. Preclassification by homomorphic equivalence
4. The universal algebraic approach
5. Summary

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Poset-SAT

Φ... finite set of quantifier-free $\{\leq\}$-formulas

Poset-SAT(Φ)

Instance:

- Variables $\{x_1, \ldots, x_n\}$ and
- finitely many formulas $\phi_i(x_{i_1}, \ldots, x_{i_k})$, where each $\phi_i \in \Phi$.

Question:

Is $\bigwedge \phi_i(x_{i_1}, \ldots, x_{i_k})$ satisfiable in a partial order (*poset*)?

Complexity of Poset-SAT(Φ) is always in NP.

Question

For which Φ is Poset-SAT(Φ) in P?

Example 1

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Poset-SAT(<)

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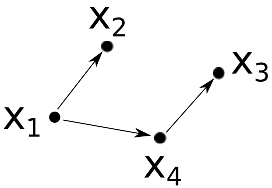

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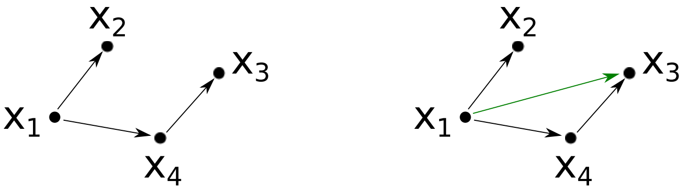

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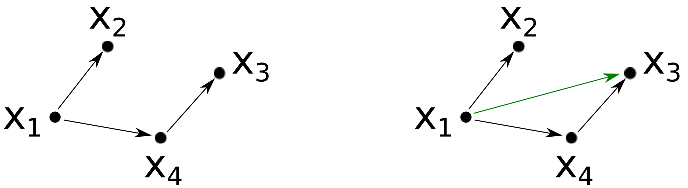

Poset-SAT($<$) is in P.

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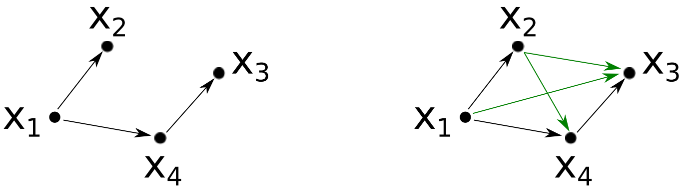
$x < y := x \leq y \wedge \neg(y \leq x).$

Temp-SAT($<$)

Instance: Variables $\{x_1, \ldots, x_n\}$ and formulas $x_{i_1} < x_{i_2}$.
Question: Is $\bigwedge(x_{i_1} < x_{i_2})$ satisfiable in a **linear** order?

Example: $x_1 < x_2$, $x_1 < x_3$, $x_3 < x_4$



Temp-SAT($<$) = Poset-SAT($<$)

Examples

In general Poset-SAT(Φ) $\neq$ Temp-SAT(Φ)!

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Example 2

$x \perp y$:= incomparability relation
$Q(x, y, z) := (x < y \vee x < z)$

Poset-SAT($\perp, Q$) is NP-complete
Temp-SAT($\perp, Q$) is in P.

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Poset-SAT($\perp, Q$) is NP-complete
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Example 3

$T(x, y, a, b) := (x < y \wedge a < b) \vee (y < x \wedge b < a) \vee (x \perp y \wedge a \perp b)$.

Poset-SAT(T) is trivial;
Temp-SAT(T) is NP-complete.

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For every $\{\leq\}$-formula $\phi(x_1, \ldots, x_n)$ we define the relation

$$R_\phi := \{(a_1, \ldots, a_n) \in P^n : \phi(a_1, \ldots, a_n)\}.$$

For a set Φ of formulas we define the structure:

$$\Gamma_\Phi := (P; (R_\phi)_{\phi \in \Phi})$$

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For a set Φ of formulas we define the structure:

$$\Gamma_\Phi := (P; (R_\phi)_{\phi \in \Phi})$$

Γ_Φ is called a reduct of $\mathbb{P}$, i.e. a structure that is first-order definable in $\mathbb{P}$.

Poset-SAT as CSP

An instance of Poset-SAT(Φ) with

- variables $\{x_1, \ldots, x_n\}$ and
- formulas $\phi_1, \ldots, \phi_k$ with $\phi_i \in \Phi$.

has a solution if and only if $\exists x_1, \ldots, x_n \, (\phi_1 \wedge \cdots \wedge \phi_k)$ holds in Γ_Φ.

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$\rightarrow$ Use nice properties of $\mathbb{P}$ for complexity classification.

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Example 3: $\mathrm{CSP}(P; T)$

$T(x, y, a, b) = (x < y \wedge a < b) \vee (y < x \wedge b < a) \vee (x \perp y \wedge a \perp b)$

There is an $g_\perp \in \mathrm{End}(P; T)$ with $g_\perp(T)$ is countable antichain. So $\mathrm{CSP}(P; T) = \mathrm{CSP}(\mathbb{N}; x \neq y \wedge a \neq b)$.

Homomorphic equivalence

Proposition (MK, Van Pham '16)

Let Γ be reduct of $\mathbb{P}$. Then:

1. $\mathrm{End}(\Gamma)$ contains a constant,
2. $\mathrm{End}(\Gamma)$ contains $g_<$ that maps P to a chain $\cong \mathbb{Q}$,
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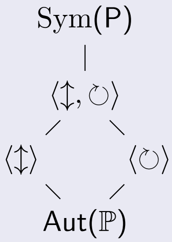
$\rightarrow$ We only need to study $\mathrm{CSP}(\Gamma)$, where $\overline{\mathrm{Aut}(\Gamma)} = \mathrm{End}(\Gamma)$.

Automorphism groups

Theorem (Pach, Pinsker, Pongrácz, Szabó '14)

Let Γ be a reduct of $\mathbb{P}$. Then $\mathrm{Aut}(\Gamma)$ is equal to one of the following:



$\updownarrow$: bijection with $x < y \leftrightarrow \updownarrow x > \updownarrow y$

$\circlearrowright$: "rotation" at a generic upwards-closed set

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For $f : \Gamma^n \to \Gamma$ we say f is a polymorphism of Γ if for all relations R of Γ: $r_1, \ldots, r_n \in R \to f(r_1, \ldots, r_n) \in R$.

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$\rightarrow$ Polymorphisms determine the complexity

Example

Let $e_{\leq} : (P; \leq)^2 \rightarrow (P; \leq)$ be an embedding:

$$e_{\leq}(x, y) \leq e_{\leq}(x', y') \Leftrightarrow x \leq x' \wedge y \leq y'$$

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Problem: How does $\mathrm{Pol}(\Gamma)$ look like? When is $e_{\leq} \in \mathrm{Pol}(\Gamma)$?

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Proof idea (very roughly):

If R not pp-definable in Γ there is a $f \in \mathrm{Pol}(\Gamma)$ violating R. Ramsey properties of $\mathbb{P}$ imply that there is a *canonical* function $g \in \mathrm{Pol}(\Gamma)$ violating R.

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$\to$ Look for relations that imply NP-hardness.
$\to$ Use canonical functions for P.

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Proposition

Let Γ be s.t. $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{P})}$. Then either

- Low is pp-definable in Γ (and $\mathrm{CSP}(\Gamma)$ is NP-c)
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Proof idea.

Assume $\mathrm{Pol}(\Gamma)$ preserves $<$, $\perp$, but violates Low.

Canonical functions

$$\mathrm{Low}(x, y, z) := (x < y \wedge z \bot xy) \vee (x < z \wedge y \bot xz).$$

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To do: show that such f generates $e_<$ or $e_\leq$. □

Outline

1. Poset-SAT problems
2. Poset-SAT as CSP over the random partial order
3. Preclassification by homomorphic equivalence
4. The universal algebraic approach
5. **Summary**

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Consequence:

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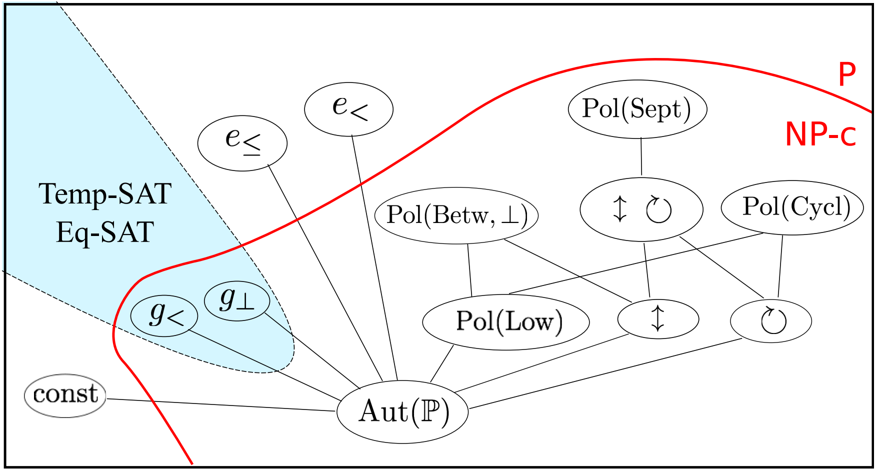
Given Φ, it is decidable to tell if Poset-SAT(Φ) is in P.

Lattice of polymorphism clones



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- or Γ is homomorphic equivalent to a Δ, such that:

$$\xi : \mathrm{Pol}(\Delta, c_1, \ldots, c_n) \to \mathbf{1}$$

  and $\mathrm{CSP}(\Gamma)$ is NP-complete.

Thank you!