Failure of local-to-global

Alexandr Kazda, Michael Kompatscher

University of Oxford
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Deciding Maltsev conditions
Definition

A (strong) Maltsev condition $\Sigma$ is a set of functional equations, e.g. $f(x, x, y) \approx f(x, y, x) \approx f(x, x, y)$

A satisfies $\Sigma$, if $A$ has a term $f^A$ such that

$\forall x, y \in A : f^A(x, x, y) = f^A(x, y, x) = f^A(x, x, y)$. 
Deciding Maltsev conditions

Definition

A (strong) Maltsev condition Σ is a set of functional equations, e.g.
f(x, x, y) ≈ f(x, y, x) ≈ f(x, x, y)

A satisfies Σ, if A has a term $f^A$ such that
$\forall x, y \in A : f^A(x, x, y) = f^A(x, y, x) = f^A(x, x, y)$.

For a fixed Maltsev condition Σ define the computational problem:

Decide(Σ)
INPUT: finite algebra $A = (A, f_1, \ldots, f_n)$
QUESTION: Does A satisfy Σ?
Why study $\text{Decide}(\Sigma)$?

- Variant of the Subpower Membership Problem?
- Test if $f(x_1, x_2, x_3) \in F(A(x_1, x_2, x_3))$ is idempotent. 
- Testing properties of algebras
- Fun with algebras / relations

The bad news (Freese, Valeriote ‘09)

In many cases $\text{Decide}(\Sigma)$ is EXPTIME-complete

- (semilattice, $\text{CD}(n)$ for $n > 3$, $\text{t}(x, \ldots, x) \approx x$, $\text{CM}$, $\text{CD}$, \ldots)
- $\text{Decide}_{\text{id}}(\Sigma)$
  - Input: finite idempotent algebra $A = (A, f_1, \ldots, f_n)$
  - Question: Does $A$ satisfy $\Sigma$?
Deciding Maltsev conditions

Why study Decide(Σ)?

- Meta-problem for CSPs?
Deciding Maltsev conditions

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→ idempotent variant

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  test if $f(x_1, x_2, x_3) \in F_A(x_1, x_2, x_3) \leq A^{A^3}$
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(semilattice, CD(n) for \( n > 3 \), \( t(x, \ldots, x) \approx x \), CM, CD,...)
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Why study \text{Decide}(\Sigma)\? 

- Meta-problem for CSPs? Input encoded as algebra 
- variant of the Subpower Membership Problem? 
  \[ \text{test if } f(x_1, x_2, x_3) \in F_A(x_1, x_2, x_3) \leq A^{A^3} \] 
- testing properties of algebras 
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**The bad news (Freese, Valeriote ’09)** 
In many cases \text{Decide}(\Sigma) is EXPTIME-complete 
(-semilattice, CD\((n)\) for \(n > 3\), \(t(x, \ldots, x) \approx x\), CM, CD,...) 

→ **idempotent variant** 

\text{Decide}^\text{id}(\Sigma) 

\textbf{Input:} finite idempotent algebra \( A = (A, f_1, \ldots, f_n) \) 

\textbf{Question:} Does \( A \) satisfy \( \Sigma \)?
Local-to-global
**Local-to-global for** \( t(x, y) \approx t(y, x) \)

\[ A \text{ has local binary symmetric terms if } \]
\[ \forall a, b \in A \exists t_{a,b} \in \text{Clo}(A): t_{a,b}(a, b) = t_{a,b}(b, a) \]
Local-to-global for $t(x, y) \approx t(y, x)$

A has local binary symmetric terms if

$$\forall a, b \in A \exists t_{a,b} \in \text{Clo}(A): t_{a,b}(a, b) = t_{a,b}(b, a)$$

$$\iff \forall a, b \in A \exists q: \begin{pmatrix} q \\ q \end{pmatrix} \in Sg_{A^2} \left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix} \right\}$$
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Claim

If A has local binary symmetric terms, then it has a (global) binary symmetric term $t(x, y) \approx t(y, x)$. 
Local-to-global for $t(x, y) \approx t(y, x)$

**A** has *local* binary symmetric terms if

$$\forall a, b \in A \exists t_{a,b} \in \text{Clo}(A): \quad t_{a,b}(a, b) = t_{a,b}(b, a)$$

$$\iff \forall a, b \in A \exists q: \quad \begin{pmatrix} q \\ q \end{pmatrix} \in S_{2}^{2} \left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ a \end{pmatrix} \right\}$$

**Claim**

If **A** has local binary symmetric terms, then it has a (global) binary symmetric term $t(x, y) \approx t(y, x)$.

**Consequence**

$\text{Decide}(t(x, y) \approx t(y, x)) \in \text{P}$. 
Assume $\mathbf{A}$ has local binary symmetric terms

Let $R \leq \mathbf{A}^4$ be a relation s.t.

\[
\begin{pmatrix} a \\ b \\ a' \\ b' \end{pmatrix}, \begin{pmatrix} b \\ a \\ b' \\ a' \end{pmatrix} \in R
\]
Assume $A$ has local binary symmetric terms.

Let $R \leq A^4$ be a relation s.t.

\[
\begin{pmatrix}
a \\
b \\
a' \\
b'
\end{pmatrix}
, \begin{pmatrix}
a \\
b \\
a' \\
b'
\end{pmatrix} \in R
\]

Then $R$ also contains

\[
\begin{pmatrix}
q \\
q \\
t(a', b') \\
t(b', a')
\end{pmatrix}
, \begin{pmatrix}
q \\
q \\
t(b', a') \\
t(a', b')
\end{pmatrix} \in R
\]
Proof idea

Assume $A$ has local binary symmetric terms

Let $R \leq A^4$ be a relation s.t.

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  b \\
  a' \\
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\end{pmatrix},
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  a \\
  b' \\
  a'
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Then $R$ also contains

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  q \\
  t(b', a') \\
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  q \\
  q' \\
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$$

induction over all pairs $(a, b)$... $A$ contains a binary cyclic term.
Failure for minority

\[ m \text{ is minority operation if } \]

\[ m(x, x, y) \approx m(x, y, x) \approx m(x, x, y) \approx y \]
Failure for minority

\( m \) is minority operation if

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m(x, x, y) \approx m(x, y, x) \approx m(x, x, y) \approx y
\]

**Theorem (Kazda, Opršal, Valeriote, Zhuk ’19)**

For every \( k \geq 2 \), \( \exists A_k \), idempotent with \( |A_k| = 4k \), s.t.

- \( A_k \) has local minority on subsets of size \( k - 1 \)
- \( A_k \) has no 'global' minority


Note: This does not prove hardness of deciding minority, and \( \text{Decide id}\text{(minority)} \in \text{NP} \).

Question: When else does local-to-global fail?
Failure for minority

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**Theorem (Kazda, Opršal, Valeriote, Zhuk ’19)**

For every $k \geq 2$, $\exists A_k$, idempotent with $|A_k| = 4k$, s.t.

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**Question**

When else does local-to-global fail?
G-terms
**G-terms**

**Definition**
For $G \leq \text{Sym}(n)$... permutation group $t$ is a \textbf{$G$-term*} if 

$t(x_1, \ldots, x_n) \approx t(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all $\pi \in G$.

*suggestions for better names are welcome.*
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- If $G = \text{Sym}(n)$: symmetric terms

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- If $G = \text{Sym}(n)$: symmetric terms
- If $G = \mathbb{Z}_n \leq \text{Sym}(n)$: cyclic terms
  \[ c(x_1, \ldots, x_{n-1}, x_n) \approx c(x_2, \ldots, x_n, x_1) \]

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- If 1-generated, e.g. $G = \langle (12)(345) \rangle$: 'cyclic loop conditions'

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- If 1-generated, e.g. \( G = \langle (12)(345) \rangle \): 'cyclic loop conditions'

Question (Valeriote)
For which \( G \) do \( G \)-terms have the local-to-global property?

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Theorem (Kazda, MK '20)
Local-to-global works if...

• $G$ has a fixpoint (trivial Maltsev condition)
• $G = Z_n \leq \text{Sym}(n)$: cyclic terms
• $G \leq \text{Sym}(|G|)$ acting on itself by left translation
• $G = \langle g \rangle$ (local on $k$ many tuples, with $k = \#\text{orbits}$)
• $G = D_n$ dihedral group for even $n$... even for non-idempotent $A_8$
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Theorem (Kazda, MK '20)
Local-to-global fails if...

- $G = \text{Sym}(n)$ for $n \geq 3$
- $G = D_n$: dihedral group for odd $n$
- $G = A_n$: alternating group for $n \geq 3$
- $\exists g \in G$ with one fixpoint and orbits of the same size otherwise.
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(even in idempotent algebras)
Construction for $G = \text{Sym}(3)$

Let $g = (1)(23)$ and $\mathbb{Z}_2 = \langle g \rangle$.

Let $T$ be a transversal of the orbits $g \curvearrowright \{1, 2, 3\}^3$.

Set $A = (\{0, 1\} \times \{1, 2, 3\} \cup \mathbb{Z}_2; t_0, t_1)$ with $t_0, t_1$ ternary
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- $t_i$ idempotent; most tuples mapped to $\mathbb{Z}_2$
Example

**Construction for** $G = \text{Sym}(3)$

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- $t_i(x_1, x_2, x_3) = x_1 + x_2 + x_3$ on $\mathbb{Z}_2$
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- $t_i$ on $(\{i\} \times \{1, 2, 3\})^3$ counts how often $g$ was applied (wrt $T$)
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- $t_i$ is symmetric elsewhere*
**Example**

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$t_i$ is symmetric everywhere but $(\{i\} \times \{1, 2, 3\})^3$. 
Example

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- $t_i$ on $(\{1 - i\} \times \{1, 2, 3\})^3$ is $id \in \mathbb{Z}_2$
- $t_i$ is symmetric elsewhere*

$t_i$ is symmetric everywhere but $(\{i\} \times \{1, 2, 3\})^3$.

Every term fails to be symmetric on $(\{0\} \times \{1, 2, 3\})^3$ or $(\{1\} \times \{1, 2, 3\})^3$. 

Where to go from here?

**Local-to-global**

- Finish the classification for $G$-terms
- When is idempotence necessary?
- Does local-to-global fail for Siggers?
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**Local-to-global**

- Finish the classification for $G$-terms
- When is idempotence necessary?
- Does local-to-global fail for Siggers?

**Decide($\Sigma$)**

- Other efficient algorithms for deciding $\Sigma$?
- Example: ’uniform’ subpower membership problem algorithm showed $\text{Decide}_{\text{id}}(\text{minority}) \in \text{NP}$ (Kazda, Opršal, Valeriote, Zhuk ’19)
- Is there a linear Maltsev condition $\Sigma$ with $\text{Decide}_{\text{id}}(\Sigma) \not\in \text{NP}$?
Thank you!