Endomorphism monoids of ω-categorical structures

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Countable, ω-cat. structures $\mathcal{A}$ and $\mathcal{B}$ are interdefinable iff

$$\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{B})$$

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- Can we reconstruct the topology of $\mathrm{Aut}(\mathcal{A})$?

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Reconstruction

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- transformation monoid
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No!

(Evans + Hewitt '90; Bodirsky + Evans + Pinsker + MK '15)

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Profinite groups are closed permutation groups where every orbits contains finitely many elements.

Example (Witt '54)

There are two separable profinite groups G, G' that are isomorphic, but not topologically isomorphic.

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Lemma (Hrushovski)

There is a oligomorphic Φ such that for every separable profinite group R there is an oligomorphic Σ_R:

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Look at finite sets. Partition the n-tuples into partition classes $P_1^n, P_2^n, \ldots P_n^n$ for all $n \geq 1$. This gives us a Fraïssé-class.

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This gives us $\Sigma/\Phi \cong^T \prod_{n\in\mathbb{N}} \mathrm{Sym}(n)$. $\square$

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The real proof deviates from the above.

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We get:

Result for monoids

$\overline{\Sigma_G}$ and $\overline{\Sigma_{G'}}$ are isomorphic, but not topologically isomorphic.

Oligomorphic clones

Observation

Let $I : \Gamma \to \Delta$ be a monoid homomorphism. If I sends constants to constants, it has a natural extension to a clone homomorphism $\mathrm{Clo}(\Gamma) \to \mathrm{Clo}(\Delta)$.

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This answers a question by Bodirsky, Pinsker and Pongrácz.

Thank you!