# Mal'cev algebras and difference clonoids

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# Mal'cev algebras

## Definition

An algebra **A**/variety $\mathcal{V}$ is **Mal'cev**, if it has a **Mal'cev term** $m$ :

$$y \approx m(y, x, x) \approx m(x, x, y).$$

## Examples

- Groups $m(x, y, z) = x \cdot y^{-1} \cdot z$
- Rings $m(x, y, z) = x - y + z$
- Boolean algebras
  $m(x, y, z) = (x \wedge y \wedge z) \vee (x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z)$
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### Contrapoint

Many basic questions are still open for finite Mal'cev algebras.

# Problem 1: Classification

$\mathrm{Clo}(\mathbf{A})$ = clone of term operations of $\mathbf{A}$

**Task**
On finite $A$, describe the lattice of Mal'cev clones.

We know it is

- finite if $|A| \leq 3$ [Bulatov '03]
- countably infinite if $|A| \geq 4$ [AMM '14, Idziak '99]

There are infinite chains, e.g.:

$$\mathrm{Clo}(\mathbb{Z}_4, +) \subset \mathrm{Clo}(\mathbb{Z}_4, +, 2x_1x_2) \subset \mathrm{Clo}(\mathbb{Z}_4, +, 2x_1x_2x_3) \subset \cdots$$

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**Question**

What about restrictions to (nice) subclasses of Mal'cev clones?

# Problem 2: Axiomatization

## Finite basis problem (Tarski)

Input: finite algebra **A**
Question: Does HSP(**A**) have a finite equational basis?

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## Question

Is the finite basis problem decidable for (subclasses) of Mal'cev algebras?

# Problem 3: Interpolation

SMP($\mathbf{A}$)

Input: $n \in \mathbb{N}$, a partial function $f\colon A^n \to A$
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**Question**

Is $\mathrm{SMP}(\mathbf{A}) \in P$ for every finite Mal'cev algebra?

# Most general known results

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*e.g. for idempotent **A**.

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$A = U \times L$ and $\forall f^{\mathbf{A}} \in \mathrm{Clo}(\mathbf{A})$:

$$f^{\mathbf{A}}\left(\begin{bmatrix} u_1 \\ l_l \end{bmatrix}, \dots, \begin{bmatrix} u_n \\ l_n \end{bmatrix}\right) = \begin{bmatrix} f^{\mathbf{U}}(u_1, \dots, u_n) \\ f^{\mathbf{L}}(l_1, \dots, l_n) + \hat{f}(u_1, \dots, u_n) \end{bmatrix},$$

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Example

$\mathbf{A} = \mathbb{Z}_4 = \mathbb{Z}_2 \otimes \mathbb{Z}_2$

$$x_1 + x_2 = \begin{bmatrix} u_1 \\ l_l \end{bmatrix} + \begin{bmatrix} u_2 \\ l_2 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 \\ l_l + l_2 + u_1 \cdot u_2 \end{bmatrix}.$$

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Question: How far is $\mathbf{U} \otimes \mathbf{L}$ from $\mathbf{U} \times \mathbf{L}$?

# The difference clonoid

$\mathbf{A} = \mathbf{U} \otimes \mathbf{L}$, Mal'cev

## Definition (difference clonoid)

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## Observation (MK, Mayr)

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- $\mathrm{Clo}(\mathbf{L}) \circ Diff(\mathbf{A}) \subseteq Diff(\mathbf{A})$

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(the two above together:) $Diff(\mathbf{A})$ is $(\mathbf{U}, \mathbf{L})$-**clonoid**

- $f \in \mathrm{Clo}(\mathbf{A}), \hat{d} \in Diff(\mathbf{A}) \Rightarrow f + \hat{d} \in \mathrm{Clo}(\mathbf{A})$.

# How does it help us? 1.Classification

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Classification of $(\mathbf{U}, \mathbf{L})$-clonoids

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## Example [Fioravanti '20]

Classification of $(\mathbb{Z}_p, \mathbb{Z}_q)$-clonoids for $p \neq q$ primes (finite)

$\rightsquigarrow$ Classification of *all* extensions of $\mathbb{Z}_p \times \mathbb{Z}_q$ (finite)

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finitely many $(\mathbf{U}, \mathbf{L})$-clonoids / infinite antichains

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## Example

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- But: Exact classification open! Minimal elements?

# How does it help us? 2. Axiomatization

Theorem [MK '25$^+$]

Let $\mathbf{A} = \mathbf{U} \otimes \mathbf{L}$, such that

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- $\mathbf{U}$ is ***strongly*** finitely based *in a quite restrictive sense*
- $\mathit{Diff}(\mathbf{A})$ is finitely based *as many-sorted algebra*

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**Conjecture**

Every finite 2-nilpotent Mal'cev algebra is finitely based.

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**Theorem [MK'24]**

Let $\mathbf{A} = \mathbf{U} \otimes \mathbf{L}$ such that

- $\mathrm{Clo}(\mathbf{U} \times \mathbf{L}) \subseteq \mathrm{Clo}(\mathbf{A})$, or
- $\mathbf{U}$ is Abelian (or supernilpotent).

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- $\mathrm{Clo}(\mathbf{U} \times \mathbf{L}) \subseteq \mathrm{Clo}(\mathbf{A})$, or
- $\mathbf{U}$ is Abelian (or supernilpotent).

$\Rightarrow$ SMP($\mathbf{A}$) reduces in polynomial time to SMP($\mathbf{U} \times \mathbf{L}$) and

**SMP($\mathit{Diff}(\mathbf{A})$)**

Input: $f : U^n \to L$ partial function
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## How does it help us? 3. Interpolation

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**Theorem [MK'24]**

SMP($\mathbf{A}$) $\in$ P, for every 2-nilpotent $\mathbf{A}$ with $|A| = pq$.

# Thank you!

Questions? Remarks? Counterexamples?