CC-circuits and the expressive power of nilpotent algebras

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SSAOS19 - Karolinka

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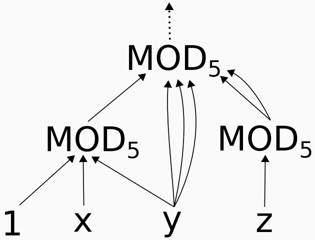

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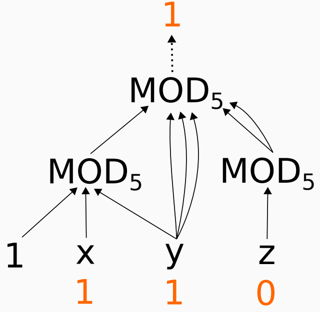

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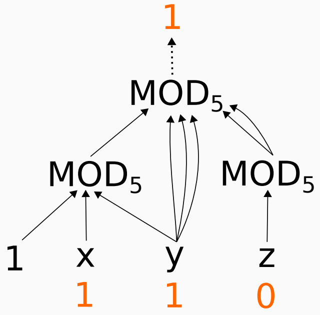

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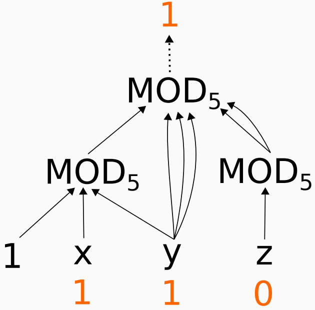

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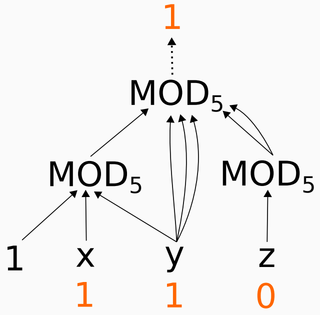

- Gates are of arbitrary fan-in
- Depth = longest path
- $CC[m]^+$-**circuit**: $\mathbb{Z}_m$-valued, also $+$-gates

A conjecture about CC-circuits

$\{\text{NEG}, \text{AND}, \text{OR}\}$-circuits of depth d need size $\Omega(e^{n^{\frac{1}{d-1}}})$ to compute $\text{MOD}_2(x_1, \ldots, x_n)$ (Håstad '87)

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- best known lower bounds in general are super-linear (CGPT '06)

Nilpotent algebras

The structure of nilpotent algebras

$\mathbf{A} = (A; f_1, \ldots, f_k)$ finite algebra
Polynomials of $\mathbf{A}$: $t(x_1, \ldots, x_n, a_1, \ldots, a_k)$: t term of $\mathbf{A}$, $a_i \in A$

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Also true for polynomials of $\mathbf{A}$

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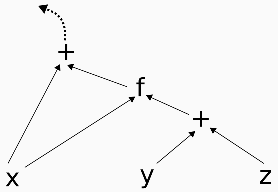

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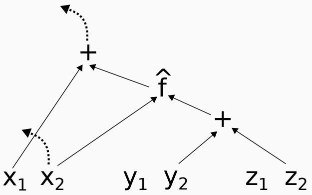

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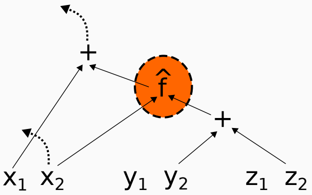

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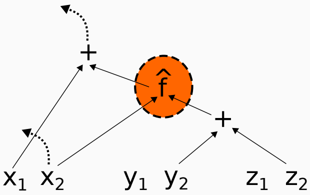

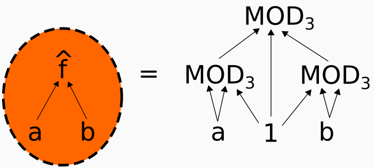

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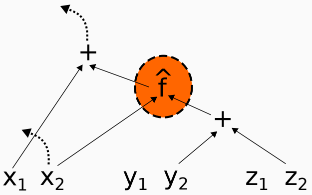

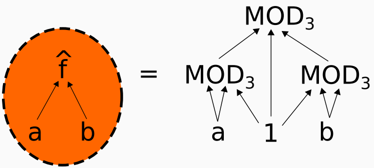

$\Rightarrow$ polynomials of $\mathbf{A}$ can be rewritten in p-time to $CC[3]^+$-circuits of depth 3

Coordinatisation of nilpotent algebras

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Theorem (Aichinger '18)

Let $\mathbf{A}$ be nilpotent, $|A| = p_1^{i_1} \cdot p_2^{i_2} \cdots p_m^{i_m}$. Then there are operations $+$,0,$-$ such that

- $(A, +, 0, -) \cong \mathbb{Z}_{p_1}^{i_1} \times \cdots \times \mathbb{Z}_{p_m}^{i_m}$
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Remark

The degree of nilpotency might increase (but $\leq \log_2(|A|)$).
E.g. $(\mathbb{Z}_4, +)$ Abelian, but $(\mathbb{Z}_4, +, +_V)$ is 2-nilpotent.

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- Every polynomial over **A** can be rewritten in polynomial time to a $CC[m]^+$-circuit of depth $\leq C(\mathbf{A})$.
- *Vice versa:* $\forall d, m$: $\exists$ nilpotent **B**, such that $CC[m]^+$-circuits of depth d can be encoded as polynomials over **B** in polynomial time.

Consequences

Conjecture (*) in nilpotent algebras

CC-circuits	in **A**...nilpotent algebra
Conjecture (*) Bounded $CC[m]$-circuits need size $\Omega(e^n)$ to compute AND. **Theorem (BST '90)** Bounded $CC[p]$-circuits cannot compute AND of arity $\geq C(d)$ **Theorem (BST '90)** Conjecture (*) is true for $m = pq$ and depth 2	

Conjecture (*) in nilpotent algebras

An operation $f : A^n \to A$ is called 0-**absorbing** iff
$f(0, x_2, \ldots, x_n) \approx f(x_1, 0, x_2, \ldots, x_n) \approx \cdots \approx f(x_1, \ldots, x_{n-1}, 0) \approx 0$.

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Conjecture (*) Bounded $CC[m]$-circuits need size $\Omega(e^n)$ to compute AND.	**Conjecture (**) (Aichinger '19)** Non-trivial absorbing polynomials of **A** of arity n have size $\Omega(e^n)$.
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Theorem (BST '90) Bounded $CC[p]$-circuits cannot compute AND of arity $\geq C(d)$	**Theorem (Aichinger, Mudrinski '10)** **A** with $\lvert A\rvert = p^k$ has only trivial absorbing polynomials of arity $\geq C(\mathbf{A})$
Theorem (BST '90) Conjecture (*) is true for $m = pq$ and depth 2	**(Idziak, Kawałek, Krzaczkowski; MK '18)** (**) is true for 2-nilpotent **A** with $\lvert A\rvert = p^k q^l$

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Question

What is the complexity for nilpotent $\mathbf{A}$?

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Assume (**) holds for **A** nilpotent.
Then CEQV(**A**) and CSAT(**A**) can be solved in $\mathcal{O}(n^{\log(n)})$.

Proof sketch:

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Note that for $|A| = p^j$: $k \leq const$
$\Rightarrow$ polynomial time algorithm. **(Aichinger, Mudrinski '10)**

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Caution! $\exists$ 2-nilpotent algebras $\mathbf{A}$ such that CEQV($\mathbf{A}$) $\in$ P, but not with 'testing' algorithm. (**Idziak, Kawałek, Krzaczkowski; MK, 18**)

Thank you!