

# The minion of free $n$ -nilpotent groups

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# Minions

Def. An (abstract) **minion**  $m$  is given by

$$m(X) \quad \forall \text{ finite set } X$$

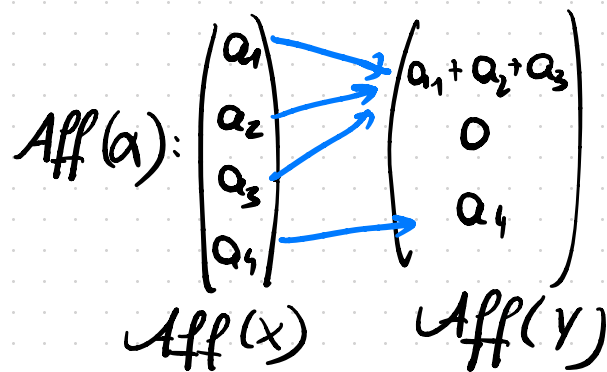
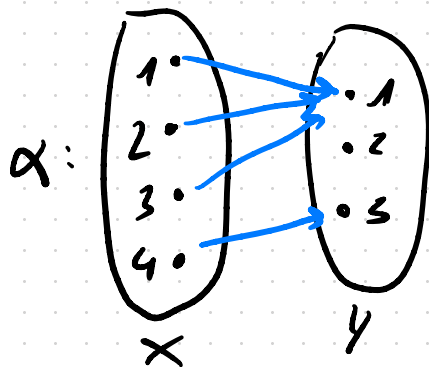
$$m(\alpha): m(X) \rightarrow m(Y) \quad \forall \alpha: X \rightarrow Y$$

$$\text{s.t. } m(\alpha \circ \beta) = m(\alpha) \circ m(\beta), \quad m(\text{id}_X) = \text{id}_{m(X)}$$

Example

$$\text{Aff}(X) =$$

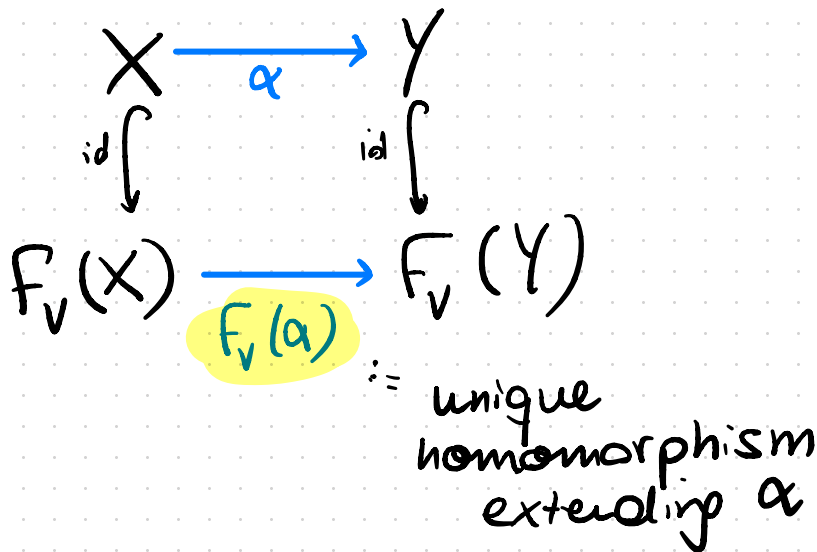
$$\{\vec{a} \in \mathbb{Z}^X \mid \sum_{i=1}^n a_i = 1\}$$



# Minions from varieties

For every variety  $V$   $\exists$  minion  $F_V$

$F_V(X) :=$  free algebra in  $V$  generated by  $X$



Remark:

$F_V$  is even an abstract clone/monad

## Question

- (multi-) linear
- does not need to reflect compositions

Can we find "nice" descriptions of  $F_V$  for:

$V = \begin{cases} \text{monoids} \\ \text{groups} \\ n\text{-nilpotent groups} \\ \text{HSP}(\underline{G}), \underline{G} \text{ finite group} \end{cases}$

$$F_V(X) = X^*$$

$$F_V(X) = F(X)$$

$$F_V(X) = F(X) / [F(X), \dots, F(X)]$$

$$F_V(X) = \text{Clo}(\underline{G})^{(X)}$$

## Motivation:

- understanding height 1 Mal'cev conditions in  $V$
- uniform algorithms for CSPs  $\iff$  minion homomorphisms

Eg.  $\text{CSP}(A)$  is solved by AIP  $\iff \text{Aff} \rightarrow \text{Pol}(A)$   
(affine integer programming)

# Minions of $\mathbb{R}$ -algebras

$\mathbb{R}$  ... comm. (semi-) ring  $X$  set

$\mathbb{R}\langle X \rangle :=$  non-commutative polynomials in  $X$  over  $\mathbb{R}$   
 $\mathbb{R}\ll X \gg :=$  " " power series " "

e.g.  $f(x, y, z) = 1 + \underline{x - 2y} + \underline{xy - x^2 - yx} + \underline{xz} \in \mathbb{Z}\langle x, y, z \rangle$

$f(x \times z) = 1 - x - x^2 + xz \in \mathbb{Z}\langle x, z \rangle$

$X \mapsto \mathbb{R}\ll X \gg$  is a minion, multi-linear (in coefficients)

Notation  $f = \sum_{w \in X^*} \langle f, w \rangle w \in \mathbb{R}\ll X \gg$

$$\mathbb{R}\ll \alpha \gg(f) = \sum_{v \in Y^+} \left( \sum_{\alpha(w)=v} \langle f, w \rangle \right) v \quad \text{for } \alpha: X \rightarrow Y$$

Can we embed our minions?

# Counting subwords

Def. The Magnus homomorphism is defined by

$$\mu: \mathbb{Z}\langle X \rangle \rightarrow \mathbb{Z}\langle X \rangle$$

$$\mu(x) = 1+x \quad \forall x \in X$$

E.g.  $\mu(xyxy) = (1+x)(1+y)(1+x)(1+y)$

$$= 1 + 2x + 2y + x^2 + y^2 + 3xy + yx + xyx + yxy + x^2y + xy^2 + xyxy$$

Def  $v, w \in X^*$   $\binom{v}{w} := \# \text{subsequences of } v \text{ equal to } w$

$$\mu(v) = \sum_{w \in X^*} \binom{v}{w} w$$

$$\binom{xyxy}{x} = 2$$

$$\binom{xyxy}{xy} = 3$$

# Combinatorics on subwords

Def. (infiltration product)  $u, v \in X^*$

"shuffles admitting identification"

//

$u \uparrow v \in R\langle X \rangle =$  sum of all infiltrations of  $u, v$

E.g.  $xy \uparrow z = xyz + xzy + zxy$

$$xy \uparrow x = xyx + xxxy + xxy + xy$$
$$= xyx + 2xxxy + xy$$

$$\binom{w}{xy} \cdot \binom{w}{x} = \binom{w}{xyx} + 2 \binom{w}{xxxy} + \binom{w}{xy} \quad \forall w \in X^*$$

In general:

$$\langle \mu(w), u \rangle \cdot \langle \mu(w), v \rangle = \langle \mu(w), u \uparrow v \rangle \quad \forall f, u, v \in X^*$$

## 1) "Subword counting" minion

Def.  $K_R(X) := \{f \in R \langle\langle X \rangle\rangle \mid \langle f, u \rangle \cdot \langle f, v \rangle = \langle f, uv \rangle \ \forall u, v \in X^*\}$

$K_R(X)$  is subminion of  $R \langle\langle X \rangle\rangle$

Theorem (Lothaire '83; Chen, Fox, Lyndon '53)

$\mu: X^* \rightarrow K_{\text{IN}}(X)$  is isomorphism for finite  $X$   
 $\Rightarrow K_{\text{IN}}$  is minion of free monoids

equationally boring minion!

$\leadsto$  Can we lift this to groups?

# The Magnus homomorphism for groups

$$\mu: X^* \rightarrow \mathcal{K}_{\mathbb{N}}(X) \subseteq \mathbb{N}\langle\langle X \rangle\rangle$$

$$x \mapsto 1+x \quad \forall x \in X$$

extends to a group embedding

$$\mu: F(X) \rightarrow \mathbb{Z}\langle\langle X \rangle\rangle^*$$

(invertible power series)

$$x^{-1} \mapsto (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

Lemma (Chen, Fox, Lyndon '58)

$$\begin{pmatrix} x^{-1} \\ x^n \end{pmatrix} = \langle \mu(x^{-1}), x^n \rangle = (-1)^n$$

$$\mu(F(X)) \subseteq \mathcal{K}_{\mathbb{Z}}(X) = \{ f \in \mathbb{Z}\langle\langle X \rangle\rangle \mid \langle f, u \rangle \cdot \langle f, v \rangle = \langle f, uv \rangle \}$$

$\forall u, v \in X^*$

but  $\neq$



## Commutators and $\mu$

E.g. For  $x, y \in X$   $[x, y] = x^{-1}y^{-1}xy$

$$\begin{aligned}\mu([x, y]) &= (1 - x + x^2 \pm \dots)(1 - y + y^2 \pm \dots)(1 + x)(1 + y) \\ &= 1 + xy - yx + \dots \quad \rightarrow \text{no linear part}\end{aligned}$$

In general:

$F_1 = F(X)$   $F_{i+1} = [F_1, F_i]$  lower central series,  $w \in X^n$

.)  $f \in F_{n+1} \Rightarrow \begin{pmatrix} f \\ w \end{pmatrix} = 0$

.)  $f, g \in F_n \Rightarrow \begin{pmatrix} f \cdot g \\ w \end{pmatrix} = \begin{pmatrix} f \\ w \end{pmatrix} + \begin{pmatrix} g \\ w \end{pmatrix}$

.)  $f \in F_r, g \in F_s, r+s=n \Rightarrow \begin{pmatrix} [f, g] \\ w \end{pmatrix} = \begin{pmatrix} f \\ w_{[1, r]} \end{pmatrix} \cdot \begin{pmatrix} g \\ w_{[r+1, n]} \end{pmatrix} - \begin{pmatrix} g \\ w_{[1, s]} \end{pmatrix} \cdot \begin{pmatrix} f \\ w_{[s+1, n]} \end{pmatrix}$

# The minion of free $n$ -nilpotent groups

$$\varphi \in F_{n+1} \Leftrightarrow \forall w \in X^n \left( \frac{f}{w} \right) = 0$$

Theorem (Chen, Fox, Lynen's 8)

$$\mu: F(X)/F_{n+1} \cong K_{\mathbb{Z}}(X)/X^n \Rightarrow$$

$$\varprojlim_n F(X)/F_{n+1} \cong K_{\mathbb{Z}}(X)$$

$K_{\mathbb{Z},n}(X) := K_{\mathbb{Z}}(X)/X^n$   
is the minion  
of free  $n$ -nilpotent groups

Example

$$K_{\mathbb{Z},2}(X) = \left\{ f \in \mathbb{Z}\langle X \rangle \mid \begin{array}{l} \deg(f) \leq 2 \\ \langle f, x \rangle \cdot \langle f, y \rangle = \langle f, xy \rangle + \langle f, yx \rangle \\ \langle f, x \rangle^2 = 2\langle f, x^2 \rangle + \langle f, x \rangle \\ \langle f, 1 \rangle = 1 \end{array} \right\}$$

## Background on the proof of CFL theorem

$$F(X)/F_{n+1} \cong K_2(X)/X^n$$

$w \in X^*$  is **Lynndon word** over  $(X, <)$

if  $|w| = 1$  or  $w = U \cdot V$   $U <_{\text{lex}} V$  **Lynndon**

e.g.  $X = \{x < y\} \rightarrow xy, xxy, xxyxy$  **Lynndon**

$C_w$  **standard commutators**

$[x, y], [x, [xy]], [[x, xy], [xy, y]]$

$\{C_w \mid |w| = n\}$  freely generates  $F_n/F_{n+1}$   $\binom{C_w}{w} = 1, \binom{C_w}{v} = 0 \quad \forall \begin{matrix} v <_{\text{lex}} w \\ |v| = |w| \end{matrix}$

Remark  $(\bigoplus_{i \geq n} F_i/F_{i+1}, \cdot, [\cdot, \cdot])$  is free Lie algebra generated by  $\{C_w \mid |w| = n\}$

What about finite nilpotent groups?

$$K_{R,n}(x) = K_R(x)/x^n$$

Corollary (also: Thérien '83)

1) Let  $\underline{G}$  be  $n$ -nilpotent  $q = \exp(\underline{G})$ :

Then  $\exists$  minion homomorphism

$$K_{\mathbb{Z}_q, n} \rightarrow \text{Clo}(\underline{G})$$

2) Let  $\underline{G}$  be finite  $p$ -group.

Then  $\exists n \leq |\underline{G}|$ :

$$K_{\mathbb{Z}_p, n} \rightarrow \text{Clo}(\underline{G})$$

⚠  $n$  can be  
bigger than  
 $\text{nil}(\underline{G})$

# Algorithmic implications

Minion

$$1) \text{Clo}(D_8) \cong \text{Clo}(Q_8) \cong K_{\mathbb{Z}_2, 2}$$

$$2) K_{\mathbb{Z}_p, p^{n-1}} \rightarrow \text{Clo}(\mathbb{Z}_{p^n})$$

(Lucas's theorem)

$$3) \underline{A} \text{ } n\text{-supernil. Mol'cev} \Rightarrow$$

$\underline{A}$  has „ $(\mathbb{Z}_K n)$ -symm. terms”

$$4) \underline{A} \text{ Taylor} \Rightarrow$$

$\underline{A}$  has „palette  $(\mathbb{Z}_K, n)$ -symm. terms”

CSP solved by

[Barto, Hodek, Zhuk '26]

$\text{CSP}(\mathbb{D}_8)$  2nd level of  $\mathbb{Z}_2$

$\text{CSP}(\mathbb{Z}_{p^2})$   $p$ th level of  $\mathbb{Z}_p$

unpublished

[Barto K, Zhuk ≥26]

$n$ -th level of AIP

[Zhuk ≥26]

$\text{CSP}(A)$  solved by  $\text{Sing}(AIP^n)$

Thank you!

Questions?

Answers?

Counterexamples?