G-terms and the local-global property

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Outline

1) G-terms Σ_G

   + examples / applications

2) Interpretability order for Σ_G

3) Deciding Σ_G with local-global property

   •) success for regular G

   •) failure for $G = Sym(n), A_n, \dots$

$\boxed{G\text{-terms}}$

let $\underline{A}$ .. algebra

$G \leq Sym(n)$ permutation group

then $t(x_1, \ldots, x_n) \in Clo(\underline{A})$ is called a

G-term if

$$\boxed{\begin{array}{l} \underline{A} \models t(x_1, \ldots, x_n) \approx t(x_{\pi(1)}, \ldots, x_{\pi(n)}) \\ \quad \text{for all } \pi \in G. \end{array}}$$

Let us write $\underline{A} \models \Sigma_G$

if $\underline{A}$ has a G-term.

Example:

$G = \langle (1,2,3) \rangle \leq Sym(3)$

t is a G-term if

$$\boxed{\begin{aligned} \underline{A} \models t(x_1 x_2 x_3) &\approx t(x_2 x_3 x_1) \\ &\approx t(x_3 x_1 x_2) \end{aligned}}$$

⚠ Σ_G depends on the group action

Examples

- Cyclic terms $C_n = \langle (1,2,\dots,n) \rangle$

  $t(x_1, x_2, \dots, x_n) \approx t(x_2, \dots, x_n, x_1)$

[BK '12]

$\mathbb{A} \models \Sigma_{C_p}$ $\forall p > |A|$ prime

$\Leftrightarrow \mathbb{A}$ is Taylor

- cyclic loop conditions $G = \langle \rho \rangle$

  e.g. $t(x_1 x_2\ x_3 x_4 x_5) \approx t(x_2 x_1\ x_4 x_5 x_3)$

  $\rho = (12)(345)$

[BSV '21]

classification $\Sigma_{\langle \rho \rangle}$ $\overset{\mathrm{Pol}(\mathbb{D})}{\longleftrightarrow}$ classification of smooth digraphs $\mathbb{D}$ up to pp-interpretation

Σ_G for minions / PCSPs

$\Sigma_G = \{ f(x_1, \dots, x_n) \approx f(x_{\pi(1)}, \dots, x_{\pi(n)}) \mid \pi \in G \}$ is of height 1

$\Rightarrow$ $\mathcal{M} \models \Sigma_G$ also well-defined for minions $\mathcal{M}$ (set of operations $A^n \to B$ closed under minors)

Example

- symmetric terms
  $G = Sym(n)$

  [PCSP(A, B) solved by BLP $\Leftrightarrow$ Pol(A, B) $\models \Sigma_{Sym(n)}$ $\forall n \in \mathbb{N}$

  [BBKO '19]

- block-symmetric terms
  $B_n = Sym(n) \times Sym(n+1)$

  [PCSP(A, B) solved by BLP+AIP $\Leftrightarrow$ Pol(A, B) $\models \Sigma_{B_n}$ for inf. many n

  [BGWŽ '20]

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Interpretability

Let us write

$$\Sigma_G \leq \Sigma_H \iff \forall \underline{A} \text{ algebra } \underline{A} \vDash \Sigma_H \Rightarrow \underline{A} \vDash \Sigma_G$$

$$\leq_{m_1} \qquad \forall \mathcal{M} \text{ minion } \mathcal{M} \vDash \Sigma_H \Rightarrow \mathcal{M} \vDash \Sigma_G$$

Question: Can we classify all conditions Σ_G up to interpretability $\leq$?

Observation:

Σ_G trivial $\iff$ G has a fixpoint i

$\forall \underline{A}: \underline{A} \vDash \Sigma_G$ $\qquad$ $\pi_i^n(x_1, \dots x_n) = x_i$ is a G-term.

Interpretability

- If $G \leq H \leq \mathrm{Sym}(n)$ $\Rightarrow$ $\Sigma_G \leq \Sigma_H$
- If $G = \alpha^{-1} H \alpha$ $\Rightarrow$ $\Sigma_G \sim \Sigma_H$
- surjective hom. $(h,\alpha): (H,m) \to (G,n)$ $\Rightarrow$ $\Sigma_G \leq \Sigma_H$

Ex. $t(x_1, x_2, x_3, x_4) \approx t(x_2, x_3, x_4, x_1)$ $\Rightarrow$ $s(y_1, y_2) := t(y_1 y_2 y_1 y_2)$

C_4-term $\qquad\qquad\qquad\qquad$ C_2-term

- Direct products

$\Sigma_{G\times H} \leq \Sigma_G \wedge \Sigma_H$ $\qquad$ (if $G \curvearrowright n$, $H \curvearrowright m$ then $G\times H \curvearrowright (n+m)$)

If $t(x_1 \dots x_n)$ $\Rightarrow$ $s(x_1 \dots x_n y_1 \dots y_m) := t(x_1 \dots x_n)$

G-term $\qquad\qquad$ is $G\times H$-term

- Wreath products

For $G \leq Sym(n)$ let $G \wr H \leq Sym(n \times m)$ be the wreath product.
$H \leq Sym(m)$

then $\Sigma_G \vee \Sigma_H \sim \Sigma_{G \wr H}$

We need composition

Example

If $t(x_1, x_2, x_3)$ C_3-term

then $t\begin{pmatrix} t(x_1, x_2, x_3), \\ t(x_4, x_5, x_6), \\ t(x_7, x_8, x_9) \end{pmatrix}$ is $C_3 \wr C_3$-term

„doubly cyclic" → useful in [AB21] as criterion for fin. tractable PCSPs

$\Sigma_G \vee \Sigma_H \leq_{h_1} \Sigma_{G \wr H}$

$\downarrow$

Partial classifications

- Cyclic terms [Olšák '18]

$$\Sigma_{C_n} \sim \bigvee_{\substack{p \mid n \\ p \text{ prime}}} \Sigma_{C_p} \qquad \Sigma_{C_n} \le \Sigma_{C_m} \Leftrightarrow \mathrm{rad}(n) \mid \mathrm{rad}(m)$$

- Cyclic loop conditions [BSV '21]
  are joins of 'prime' CLC

Ex. $\Sigma_{(4,6,7)} \sim \Sigma_{(2,2,7)} \vee \Sigma_{(2,3,7)} \sim \Sigma_{(2,7)} \vee \Sigma_{(2,3,7)} \sim \Sigma_{(2,7)}$

$$\Sigma_{(n_1 \dots n_\ell)} \le \Sigma_{(m_1, \dots, m_k)} \Leftrightarrow \forall m_i \, \exists n_j : \mathrm{rad}(n_j) \mid \mathrm{rad}(m_i)$$

New results [KK '21]

p-groups: If G is a p-group then $\Sigma_G \leq \Sigma_{C_p}$.

Regular groups: $G \leq Sym(G)$ with $g(h) = g \circ h$

- if $K \simeq G \times H$ (as abstract groups) $\Rightarrow \Sigma_K \sim \Sigma_G \vee \Sigma_H$

  with reg. action!

- Corollary: If G regular & nilpotent then $\Sigma_G = \bigvee_{p \| G|} \Sigma_{C_p}$

To do: Fully classify p-regular- $\leq Sym(n)$ - groups.

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Deciding Maltsev conditions

Decide (Σ)

INPUT: Finite $\underline{A} = (A, f_1, \ldots, f_n)$

QUESTION: Does $\underline{A} \models \Sigma$?

Question (Valeriote): what is the complexity of Decide (Σ_G)?

Σ	all $\underline{A}$	idempotent $\underline{A}$
•) $m(yxx) \approx m(xxy) \approx y$	EXP-C	P
•) $w(yx\ldots x) \approx \ldots \approx w(x\ldots xy) \approx y$	EXP-C	P
•) $f(area) \approx f(rare)$	P	P
•) semilattice	EXP-C	EXP-C
•) $t(x_1 \ldots x_n) \approx t(x_2 \ldots x_n x_1)$	P	P
•) $m(yxx) = m(xyx) = m(xxy) = y$	?	NP

local-global

1-Local-global for Σ_{C_n}

[BKMMN '09]

Claim $\underline{A} \models f(xy) \approx f(yx) \Leftrightarrow$

$\forall a = (a_1, a_2) \in A^2 \ \exists f_a \in \mathrm{Clo}\,\underline{A} : \ f_a(a_1 a_2) = f_a(a_2 a_1)$

Proof sketch

$d_1 = f_a(b_1 b_2)$
$d_2 = f_a(b_2 b_1)$

x	y	$f_a(xy)$	$f_a(yx)$	$f_d(f_a(xy), f_a(yx))$
a_1	a_2	c	c	c'
a_2	a_1	c	c	c'
b_1	b_2	d_1	d_2	e
b_2	b_1	d_2	d_1	e
$\vdots$		$\vdots$		$\vdots$

$\square$

$\binom{c}{c} \overset{?}{\in} Sg_{\underline{A}^2}\begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$ decidable in $O(\mathrm{ar}(\underline{A}) \cdot |A|^2)$

$\Rightarrow$ Decide(Σ_{C_2}) can be solved in $O(\mathrm{ar}(\underline{A}) \cdot |A|^4)$

$\boxed{n\text{-local-global property}}$

Σ ... h1 identities in variables $x_1 \dots x_k$, function symb. $f_1, f_2 \dots f_\ell$, m many different minors

$\underline{A}$ algebra $F \subseteq A^k$

then $\underline{A} \models_F \Sigma$ if $\exists f_1^F, f_2^F, \dots, f_\ell^F$ which satisfy Σ for $(x_1 x_2 \dots x_k) \in F$.

<u>Def</u>. Σ has the <u>n-local-global property</u> if

$$\forall \underline{A}: \quad \underset{\text{for all } |F| = n}{\underline{A} \models_F \Sigma} \Rightarrow \underset{\text{for all } |F| < \infty}{\underline{A} \models_F \Sigma}$$

In this case $\mathrm{Decide}(\Sigma)$ solvable in $O\left(\mathrm{ar}(\underline{A}) \cdot |A|^{n \cdot (m+k)}\right)$

Theorem [KK'21]

$G = \underbrace{G_1}_{\text{regular}} \times \cdots \times \underbrace{G_n} \leq \mathrm{Sym}(G_1 \dot\cup \cdots \dot\cup G_n)$

for $n = 1$
proof as for Σ_{C_k}

$\Rightarrow \Sigma_G$ has $\underline{n}$-local-global property

$\mathrm{DECIDE}(\Sigma_G) \in P$

Question:
Can we generalize this result to
$\Sigma_{G \times H}$ s.t. Σ_G l.-g.
Σ_H l.-g.

Corollary [KK'21]

$\mathrm{DECIDE}(\Sigma_{\langle S \rangle}) \in P$

for cyclic loop terms.

Failure of local-global

Theorem [KK '21]

Let $G \leq \mathrm{Sym}(n)$ s.t.

- Σ_G non-trivial
- $\exists g \in G : g = (\overset{1}{\cdot})(\overset{\ell}{—})(\overset{\ell}{—})\cdots(\overset{\ell}{—})$

Then Σ_G does not have the k-local-global property for any $k \in \mathbb{N}$.

Example: for

- $G = \mathrm{Sym}(n)$ $n \geq 3$
- $G = A_n$ $n \geq 3$
- $G = D_{2n}$ n odd
- $G = \mathrm{Sym}(n) \times \mathrm{Sym}(n+1)$ $n \geq 2$

Σ_G has not local-global property.

Construction $\Sigma_{Sym(3)}$ has not 1-local-global property

let $g = (1)(23)$ $\langle g \rangle \cong \mathbb{Z}_2$

$\langle g \rangle \curvearrowright \{1,2,3\}^3$

Transversal T

1 2 3	1 1 3	3 2 2	- -
1 3 2	1 1 2	2 3 3	- -

$\underline{A} = (\{0,1\} \times \{123\} \cup \mathbb{Z}_2 ; f_0, f_1)$

•) f_i ternary, idempotent

for $\bar{t} \in T$:

•) $f_i(\{i\} \times g^{\ell}(\bar{t})) = \ell \in \mathbb{Z}_2$

•) $f_i(\{1-i\} \times g^{\ell}(\bar{t})) = 0 \in \mathbb{Z}_2$

•) $f_i(x_1, x_2, x_3) = x_1 + x_2 + x_3$ if $x_i \in \mathbb{Z}_2$

and symmetric elsewhere*

f_i symmetric everywhere except $\{i\} \times \{123\}^3$

⋮

same for every $t \in \mathrm{Clo}\, \underline{A}$.

Open questions

- What is the complexity of $\mathrm{Decide}(\Sigma_{sym(n)})$?
  - Is $\mathrm{Decide}(\Sigma_{sym(n)}) \in NP$?
  - Is there any h1 Σ with $\mathrm{Decide}(\Sigma) \notin NP$?
- Are there h1 conditions $\Sigma \sim \Sigma'$ such that Σ has l.-g. property but Σ' not?
- For Σ h1 condition: Is

  $\mathrm{Decide}(\Sigma) \sim_p \mathrm{Decide}^{\mathrm{idemp.}}(\Sigma)$?

Thank you!