Topology is relevant

in studying the complexity of infinite CSPs

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17/01/2020

JMM2020 - Denver

Local identities are relevant

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The constraint satisfaction problem

$\mathbb{A} = (A; R_1, R_2, \ldots, R_n)$... relational structure (finite language)

Definition CSP($\mathbb{A}$)
INPUT: finite relational structure $\mathbb{X}$
QUESTION: $\exists$ homomorphism $\mathbb{X} \to \mathbb{A}$?

Examples:

- CSP($K_3; E$) $\rightsquigarrow$ 3-colorability
- CSP($\{0,1\}; 0, 1, \leq$) $\rightsquigarrow$ (non-)reachability
- CSP($\mathbb{Q}; <$) $\rightsquigarrow$ acyclicity

For finite $A \rightsquigarrow$ CSP($\mathbb{A}$) $\in$ NP

The universal algebraic approach

For *finite A*:

- CSP($\mathbb{A}$) is determined by the *polymorphism clone*:
  $\text{Pol}(\mathbb{A}) = \{f\colon \mathbb{A}^n \to \mathbb{A} : f \text{ homomorphism}, n \geq 1\}$
- If Pol($\mathbb{A}$) satisfies only *trivial* h1 identities, then CSP($\mathbb{A}$) $\in$ NP-c
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trivial = satisfiable by projections
h1 = no composition is used

Examples

- $f(f(x,y),z) \approx f(x,f(y,z))$: not h1, trivial ($f(x,y) = x$)
- $f(x,y,x,z,y,z) \approx f(y,x,z,x,z,y)$: h1, non-trivial

What about 'nice' infinite domain CSPs?

ω-categorical structures

Definition

A structure $\mathbb{A}$ is ω-**categorical** if
$\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ has finitely many orbits $\forall n \geq 1$.

E.g.: $(\mathbb{Q}, <)$, ctb. free Boolean algebra, random graph, Henson digraphs...

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- **Dichotomy conjecture** If $\mathbb{A}$ is ω-categorical + reduct of a finitely bounded homogeneous structure and Pol($\mathbb{A}$) satisfies *non-trivial* h1 identities on every finite $F \subseteq A$, then CSP($\mathbb{A}$) $\in$ P

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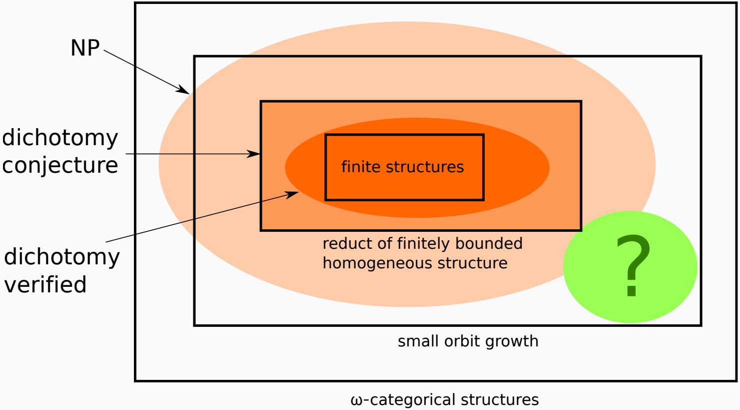
The range of the conjecture



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small orbit growth... number of orbits $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ is $o(2^{2^n})$.

Question: How wild is (?):

- local vs global h1 identities?
- Complexity of CSP($\mathbb{A}$)?

Results

Theorem (BMOOPW ’19)

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New result 2 (GJMKP '20)

There exist **finite** language structures $\mathbb{A}$ of small orbit growth with Pol($\mathbb{A}$) satisfying non-trivial h1 identities, such that CSP($\mathbb{A}$) can be

- $L \leq \text{CSP}(\mathbb{A}) \leq \text{coNP}^L$ for arbitrary L
- coNP-complete
- coNP-intermediate

That's it

Thank you!