The equivalence problem for nilpotent algebras

in congruence modular varieties

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Charles University Prague

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The equivalence problem

The equivalence problem for finite algebras

$\mathbf{A} = (A, f_1, \ldots, f_n)$... finite algebra

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Circuit equivalence problem CEQV($\mathbf{A}$)

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QUESTION: Does $\mathbf{A} \models p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$?

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$\mathrm{CEQV}(\mathbf{A}) \in \mathrm{coNP}$

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Main question

What are criteria for tractability (P) or hardness (coNP-c)?

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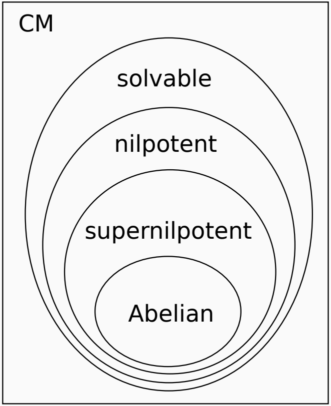
- (encoding not relevant in nilpotent case)

In congruence modular varieties

A... from congruence modular variety



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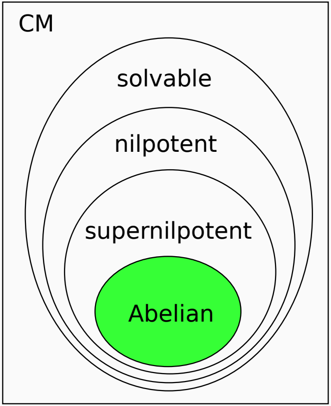

- **A** Abelian $\leftrightarrow$ module. CEQV(**A**) $\in$ P

  compute normal form $p(\bar{x}) \approx \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i$

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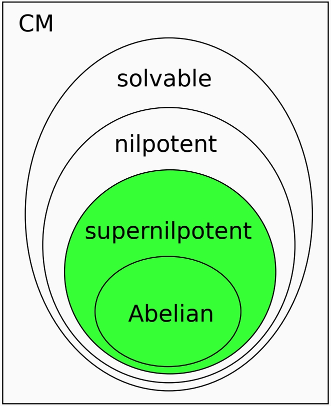

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  $p(x_1 \ldots, x_n) \approx 0$ iff $p(a_1 \ldots, a_n) = 0$, for all $\bar{a}$ with at most k-many $a_i \neq 0$
  (Aichinger, Mudrinski '10)

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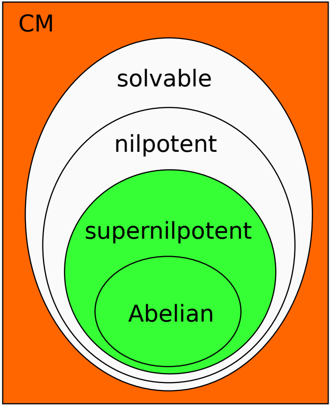

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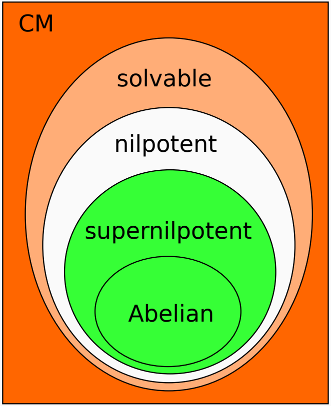

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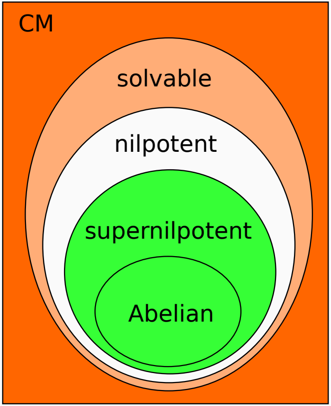

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- **A nilpotent, not supernilpotent...?**
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Nilpotent algebras

The structure of nilpotent algebras

A... n-nilpotent from CM variety.

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Theorem (Freese, McKenzie)

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Then $\exists$ **L** Abelian, **U** is $(n-1)$-nilpotent, $A = L \times U$ and

$$f^{\mathbf{A}}((l_1, u_1), \ldots, (l_n, u_n)) = (f^{\mathbf{L}}(l_1, \ldots, l_n) + \hat{f}(u_1, \ldots, u_n), f^{\mathbf{U}}(u_1, \ldots, u_n)),$$

for all operations.

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Checking $\mathbf{A} \models p^{\mathbf{A}}(x_1, \ldots, x_n) \approx 0$ is equivalent to checking
$p^{\mathbf{U}}(u_1, \ldots, u_n) \approx 0$ in **U**
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we need to analyze the expressions $\hat{p}$!
Operations $\hat{p}\colon U^n \to L$ form an (**L**)-clonoid.

Example 1 ($|L|$ and $|U|$ coprime)

$$\mathbf{L} \otimes^T \mathbf{U} = (\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f) \text{ with } p \neq q,\ \hat{f}(u) = \begin{cases} 1 \text{ if } u = 0 \\ 0 \text{ else} \end{cases}$$

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Simplify $\hat{p}$ by:

- $\hat{f}(u) \approx \hat{f}(2u) \approx \cdots \approx \hat{f}((q-1)u)$
- $1 \approx \sum_{i=0}^{p-1} \hat{f}(u - i)$
- axioms for **L** and **U** (e.g. $p \cdot \hat{f}(u) \approx 0$, $\hat{f}(u + q \cdot u') \approx \hat{f}(u)$)

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$\rightarrow$ compute in **polynomial time** the representation:

$$\hat{p}(u_1, \ldots, u_n) \approx \gamma_0 + \sum \gamma_{\bar{\delta}} \cdot \hat{f}(1 + \sum \delta_i \cdot u_i)$$

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This is representation is **unique:**
$\{\hat{f}(1 + \sum_{i=1}^n \delta_i \cdot u_i)\} \cup \{1\}$ is a basis of the vector space $\mathbf{L}^{U^n}$ for every n.

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Thus $\mathrm{CEQV}((\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f)) \in \mathrm{P}$:
To check $p(x_1, \dots, x_n) \approx 0$ compute normal form of p, and check if $= 0$.

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Observation 2

- Only finitely many identities used to compute normal form.

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To check $p(x_1, \ldots, x_n) \approx 0$ compute normal form of p, and check if $= 0$.

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- *all* operations $U^n \to L$ are generated by $\hat{f}$.

$\Rightarrow$ For every 2-nilpotent $\mathbf{A}$ with $\mathbf{L} = \mathbb{Z}_p$, $\mathbf{U} = \mathbb{Z}_q$:
$\mathrm{Pol}(\mathbf{A}) \leq \mathrm{Pol}((\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f))$

$\Rightarrow$ $\mathrm{CEQV}(\mathbf{A}) \in \mathrm{P}$.

Observation 2

- Only finitely many identities used to compute normal form.

$\Rightarrow$ $(\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f)$ is finitely based.

Example 2 ($|L|$ and $|U|$ not coprime)

Let $\mathbf{L} \otimes^T \mathbf{U} = (\mathbb{Z}_p \times \mathbb{Z}_p, +, (0,0), -, f)$, f binary

$$\hat{f}(u_1, u_2) = \begin{cases} 1 \text{ if } u_1 = u_2 = 0 \\ 0 \text{ else} \end{cases}$$

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Remark: All 2-supernilpotent algebras with $\mathbf{L} = \mathbf{U} = \mathbb{Z}_p$ reduce to this one.

CEQV **for 2-nilpotent algebras**

2-nilpotent algebras

Theorem (Kawałek, MK, Krzaczkowski ’19)

Let $\mathbf{A} = \mathbf{L} \otimes^{T} \mathbf{U}$ 2-nilpotent. Then $\mathrm{CEQV}(\mathbf{A}) \in \mathrm{P}$

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5. **Major problem:** Non vector-spaces, e.g. $\mathbf{U} \cong \mathbb{Z}_9$ $\mathbf{L} \cong \mathbb{Z}_4$. How to find normal form of $\mathbf{L}^{U^n}$? $\to$ **different approach**

A different approach

Example: $\mathbf{U} = \mathbb{Z}_9$ $\mathbf{L} = \mathbb{Z}_4$.
Let $\hat{f}(u) = 1$ if $u = 0$ and $\hat{f}(u) = 0$ else.

- Check $\hat{f}(u+1) + \hat{f}(u+v) + \hat{f}(u+4v) \approx const$

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- How to generalize to $n-1$-nilpotent **U**?

A very helpful extension

Aichinger coordinatisation

Proposition (Aichinger '18)

Let **A** be nilpotent, $|A| = p_1^{i_1} \cdot p_2^{i_2} \cdots p_m^{i_m}$. Then there are operations $+, 0, -$ such that

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Remark

The degree of nilpotence might increase (but $\leq \log_2(|A|)$).
E.g. $(\mathbb{Z}_4, +)$ Abelian, but $(\mathbb{Z}_4, +, +_V)$ is 2-nilpotent.

Summary

A... *n*-nilpotent, extension of a group $\mathbb{Z}_{p_1}^{i_1} \times \cdots \times \mathbb{Z}_{p_m}^{i_m}$

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By induction on n:

Theorem (MK)

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Open questions

A... finite nilpotent, from a CM variety

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Question

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Is **A** *itself* finitely based?

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Is the equation/circuit solvability problem of **A** in P?

The end

Thank you!