The equivalence problem for nilpotent algebras

in congruence modular varieties

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Charles University Prague

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AAA98 - Dresden

The equivalence problem

The equivalence problem for finite algebras

$\mathbf{A} = (A, f_1, \ldots, f_n)$... finite algebra

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Circuit equivalence problem CEQV($\mathbf{A}$)

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QUESTION: Does $\mathbf{A} \models p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$?

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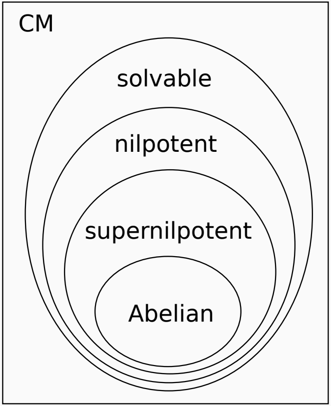
- (will set aside in this talk)

In congruence modular varieties

A... from congruence modular variety



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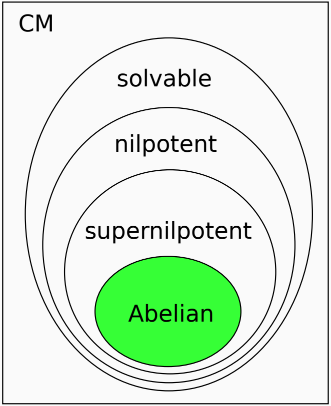

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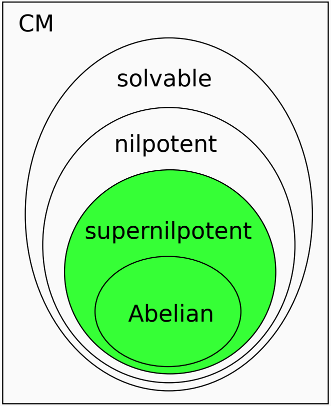

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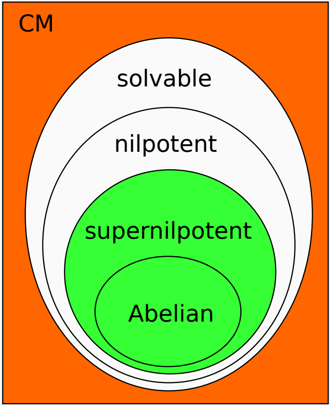

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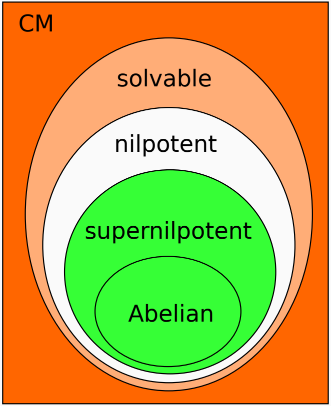

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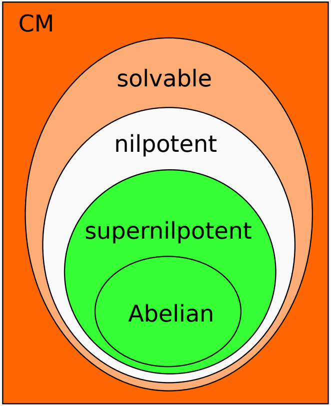

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Nilpotent algebras

The structure of nilpotent algebras

A... *n*-nilpotent from CM variety.

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Then $\exists$ **L** Abelian, **U** is $(n-1)$-nilpotent, $A = L \times U$ and

$$f^{\mathbf{A}}((l_1, u_1), \dots, (l_n, u_n)) = (f^{\mathbf{L}}(l_1, \dots, l_n) + \hat{f}(u_1, \dots, u_n), f^{\mathbf{U}}(u_1, \dots, u_n)),$$

for all operations.

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$p^{\mathbf{U}}(u_1, \ldots, u_n) \approx 0$ in **U**
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- we need to analyze the expressions $\hat{p}$!
- Operations $\hat{p}\colon U^n \to L$ form a $(\mathbf{U}, \mathbf{L})$-clonoid.

Example 1 ($|L|$ and $|U|$ coprime)

$$\mathbf{L} \otimes^T \mathbf{U} = (\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f) \text{ with } p \neq q,\ \hat{f}(u) = \begin{cases} 1 \text{ if } u = 0 \\ 0 \text{ else} \end{cases}$$

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$\rightarrow$ compute in **polynomial time** the representation:

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This is representation is **unique:**
$\{\hat{f}(1 + \sum_{i=1}^{n} \delta_i \cdot u_i)\} \cup \{1\}$ is a basis of the vector space $\mathbf{L}^{U^n}$ for every n.

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Thus $\mathrm{CEQV}((\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f)) \in \mathrm{P}$

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Observation 2

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- **Question:** Can we find such canonical extension for every $\mathbf{L}$, $\mathbf{U}$?

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- Only finitely many identities used to compute normal form.

Example 1 ($|L|$ and $|U|$ coprime)

Thus $\mathrm{CEQV}((\mathbb{Z}_p \times \mathbb{Z}_q, +, (0,0), -, f)) \in \mathrm{P}$
(compute normal form of p, and check if $= 0$)

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Example 2 ($|L|$ and $|U|$ of same characteristic)

Let $\mathbf{L} \otimes^T \mathbf{U} = (\mathbb{Z}_p \times \mathbb{Z}_p, +, (0,0), -, f)$,
$i : U \to L$ isomorphism
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Observe: All **A** with $\mathbf{L} = \mathbf{U} = \mathbb{Z}_p$ and **unary** operations reduce to this one. Analogous for *n*-**ary** operations.

CEQV **for 2-nilpotent algebras**

2-nilpotent algebras

Theorem (Kawałek, MK, Krzaczkowski '19)

Let $\mathbf{A} = \mathbf{L} \otimes^{T} \mathbf{U}$ 2-nilpotent. Then $\mathrm{CEQV}(\mathbf{A}) \in \mathrm{P}$

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Problem

Specific for abelian $\mathbf{U}$. Are we stuck in general?

AAA (Aichinger's awesome augmentations)

Coordinatisation of nilpotent algebras

Proposition (Aichinger '18)

Let **A** be nilpotent, $|A| = p_1^{i_1} \cdot p_2^{i_2} \cdots p_m^{i_m}$. Then there are operations $+$,0,$-$ such that

- $(A, +, 0, -) \cong \mathbb{Z}_{p_1}^{i_1} \times \cdots \times \mathbb{Z}_{p_m}^{i_m}$
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Remark

The degree of nilpotency might increase (but $\leq \log_2(|A|)$).
E.g. $(\mathbb{Z}_4, +)$ Abelian, but $(\mathbb{Z}_4, +, +_V)$ is 2-nilpotent.

Outline

A... *n*-nilpotent, extension of a group $\mathbb{Z}_{p_1}^{i_1} \times \cdots \times \mathbb{Z}_{p_m}^{i_m}$

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If **U** is abelian (4) can be done in P. But in general?

Example

Example (simplified)

$\mathbf{A} = (\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_5), +, f_2, f_3)$, with

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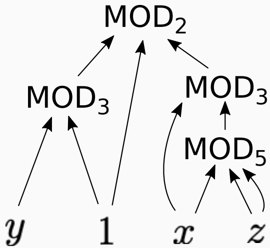

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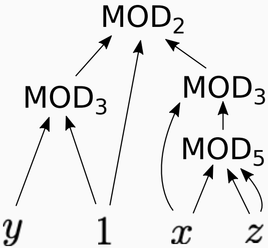

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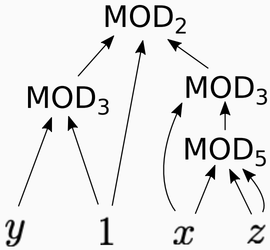

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- CEQV(**A**) reduces to check if $CC[30]$-circuits of depth 3 are ≈ 0

Open questions

A... finite nilpotent, from a CM variety

Proposition (MK)

CEQV(**A**) can be reduced to checking equivalence of $CC[n]$ circuits of depth at most k, for some n, k (and vice versa).

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Question 2

A has a finitely based nilpotent extension. Is **A** *itself* finitely based?

Thank you!