Solving equations in groups extended by their commutator

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AAA97 Wien - 03/03/2019

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Solving equations in groups

The equation solvability problem for groups

$(G, \cdot)$... finite group

Equation solvability $\mathrm{Eq}(G, \cdot)$

INPUT: A polynomial $f(x_1, \ldots, x_n)$ over G

QUESTION: Does $f(x_1, \ldots, x_n) = 0$ have a solution in G?

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Question

What are criteria for tractability (P) or hardness (NP-c / coNP-c)?

In groups

Example

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The equation solvability problem in solvable groups is decidable in

- polynomial time
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- polynomial time
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- quasipolynomial time (open conjecture about CC^0-circuits)

Adding the commutator

The complexity is sensitive to the signature!

Example A_4. (Horváth, Szabó '12)

$\mathrm{Eq}(A_4, \cdot) \in \mathrm{P}$ but adding $[x, y] = x^{-1}y^{-1}xy$:

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Proof idea: Encode 3-COLOR

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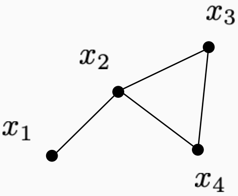

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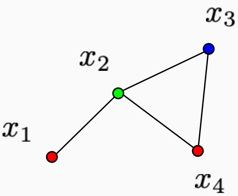

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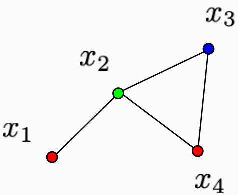

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□

Similar: p-COLOR in $G = \mathbb{Z}_p \ltimes (\mathbb{Z}_q^n)$.

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Theorem (Horváth, Szabó '11)

Every non-nilpotent G has an extension by some term $t(x_1, \ldots, x_n)$ such that $\mathrm{Eq}(G, \cdot, t(x_1, \ldots, x_n)) \in$NP-c and $\mathrm{Id}(G, \cdot, t(x_1, \ldots, x_n)) \in$coNP-c.

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$\rightarrow$ can one always choose t to be the commutator?

Reducing to 'A_4-like' groups

Verbal subgroups

A subgroup $V \leq G$ is verbal if $V = t(G, G, \ldots, G)$ for some term t. E.g. G' is verbal: $[x_1, x_2] \cdot \cdots \cdot [x_{n-1}, x_n]$.

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For $V \leq G$ verbal:

$\mathrm{Eq}(V, \cdot, [\cdot, \cdot]) \leq_p \mathrm{Eq}(G, \cdot, [\cdot, \cdot])$, $\mathrm{Id}(V, \cdot, [\cdot, \cdot]) \leq_p \mathrm{Id}(G, \cdot, [\cdot, \cdot])$

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For $V \leq G$ verbal, normal

- $\mathrm{Eq}(G/V, \cdot, [\cdot, \cdot]) \leq_p \mathrm{Eq}(G, \cdot, [\cdot, \cdot])$
- $\mathrm{Id}(G/C_G(V), \cdot, [\cdot, \cdot]) \leq_p \mathrm{Id}(G, \cdot, [\cdot, \cdot])$

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$\rightsquigarrow$ obtain a reduction of some non-nilpotent $\mathbb{Z}_p \ltimes (\mathbb{Z}_q^n)$ to G.

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Theorem (MK '18)
If $G' \leq F(G) < G$ and $\exp(G/F(G)) > 2$ then
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By our trick we can only encode 2-COLOR in $\mathrm{Eq}(\mathbb{Z}_2 \ltimes \mathbb{Z}_p, \cdot, [\cdot,\cdot])$.

But for $w(y, x_1, x_2, x_3) = y^8[[[y, x_1], x_2], x_3]$:

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INPUT: Affine subspaces $A_1, \ldots, A_k \leq \mathbb{Z}_2^n$
QUESTION: Is there an $\bar{x} \in \mathbb{Z}_2^n$ that is covered $m \cdot p$ many spaces?