Equation solvability over algebras in congruence modular varieties

Michael Kompatscher

AAA95 Bratislava - February 8, 2018

Charles University Prague

The equation solvability problem

The equation solvability problem

A... algebra

Equation solvability Eq(**A**)

INPUT: Two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ over **A**

QUESTION: $\exists a_1, \ldots, a_n \in \mathbf{A}$ such that $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$?

The equation solvability problem

A... algebra

Equation solvability Eq(**A**)

INPUT: Two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ over **A**

QUESTION: $\exists a_1, \ldots, a_n \in \mathbf{A}$ such that $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$?

Many natural problems can be phrased as Eq(**A**).

The equation solvability problem

$\mathbf{A}$... algebra

Equation solvability $\mathrm{Eq}(\mathbf{A})$

$\textsc{Input}$: Two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ over $\mathbf{A}$

$\textsc{Question}$: $\exists a_1, \ldots, a_n \in \mathbf{A}$ such that $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$?

Many natural problems can be phrased as $\mathrm{Eq}(\mathbf{A})$.

- $\mathrm{Eq}(\mathbb{Z}; +, \cdot)$: Diophantine problem

The equation solvability problem

A... algebra

Equation solvability Eq(**A**)

INPUT: Two polynomials $f(x_1, \dots, x_n), g(x_1, \dots, x_n)$ over **A**

QUESTION: $\exists a_1, \dots, a_n \in \mathbf{A}$ such that $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$?

Many natural problems can be phrased as Eq(**A**).

- $\mathrm{Eq}(\mathbb{Z}; +, \cdot)$: Diophantine problem
- $\mathrm{Eq}(\{0,1\}; \wedge, \vee, \neg)$: Boolean SAT

The equation solvability problem

A... algebra

Equation solvability Eq(**A**)

INPUT: Two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ over **A**

QUESTION: $\exists a_1, \ldots, a_n \in \mathbf{A}$ such that $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$?

Many natural problems can be phrased as Eq(**A**).

- Eq($\mathbb{Z}; +, \cdot$): Diophantine problem
- Eq($\{0, 1\}; \wedge, \vee, \neg$): Boolean SAT
- all CSPs

The equation solvability problem

A... algebra

Equation solvability Eq(**A**)

INPUT: Two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ over **A**
QUESTION: $\exists a_1, \ldots, a_n \in \mathbf{A}$ such that $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$?

Many natural problems can be phrased as Eq(**A**).

- $\mathrm{Eq}(\mathbb{Z}; +, \cdot)$: Diophantine problem
- $\mathrm{Eq}(\{0, 1\}; \wedge, \vee, \neg)$: Boolean SAT
- all CSPs

Main question

For given **A**, what is the computational complexity of Eq(**A**)?

The equation solvability problem

A... algebra

Equation solvability Eq(**A**)

INPUT: Two polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)$ over **A**

QUESTION: $\exists a_1, \ldots, a_n \in \mathbf{A}$ such that $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$?

Many natural problems can be phrased as Eq(**A**).

- Eq($\mathbb{Z}; +, \cdot$): Diophantine problem
- Eq($\{0,1\}; \wedge, \vee, \neg$): Boolean SAT
- all CSPs

Main question

For given **A**, what is the computational complexity of Eq(**A**)?

For finite algebras: Eq(**A**) $\in$ NP

Are there nice criteria for being in P or NP-complete?

The bad news

Is universal algebra the right tool to study the complexity?

The bad news

Is universal algebra the right tool to study the complexity?

Complexity not preserved by equivalence. (Horváth, Szabó ’12)

$\mathrm{Eq}(A_4; \cdot, e, ^{-1}) \in \mathrm{P}$ but adding the commutator $[x, y] = x^{-1}y^{-1}xy$:
$\mathrm{Eq}(A_4; \cdot, e, ^{-1}, [., .]) \in \mathrm{NP}$-c.

The bad news

Is universal algebra the right tool to study the complexity?

Complexity not preserved by equivalence. (Horváth, Szabó '12)

$\mathrm{Eq}(A_4; \cdot, e, ^{-1}) \in \mathrm{P}$ but adding the commutator $[x, y] = x^{-1}y^{-1}xy$:
$\mathrm{Eq}(A_4; \cdot, e, ^{-1}, [., .]) \in$ NP-c.

Possible fix

The bad news

Is universal algebra the right tool to study the complexity?

Complexity not preserved by equivalence. (Horváth, Szabó '12)

$\mathrm{Eq}(A_4; \cdot, e, ^{-1}) \in \mathrm{P}$ but adding the commutator $[x, y] = x^{-1}y^{-1}xy$:
$\mathrm{Eq}(A_4; \cdot, e, ^{-1}, [., .]) \in \mathrm{NP\text{-}c}$.

Possible fix

- investigate the 'hardest' polynomial extensions of $\mathbf{A}$

The bad news

Is universal algebra the right tool to study the complexity?

Complexity not preserved by equivalence. (Horváth, Szabó '12)

$\mathrm{Eq}(A_4; \cdot, e, ^{-1}) \in \mathrm{P}$ but adding the commutator $[x, y] = x^{-1}y^{-1}xy$:
$\mathrm{Eq}(A_4; \cdot, e, ^{-1}, [.,.]) \in$ NP-c.

Possible fix

- investigate the 'hardest' polynomial extensions of $\mathbf{A}$
- *Circuit satisfiability* $\mathrm{CSat}(\mathbf{A})$: terms are given by algebraic circuits

The bad news

Is universal algebra the right tool to study the complexity?

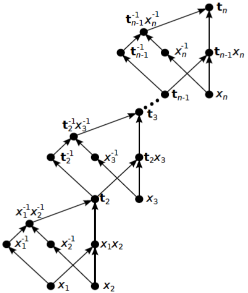
Complexity not preserved by equivalence. (Horváth, Szabó '12)

$\mathrm{Eq}(A_4; \cdot, e, ^{-1}) \in \mathrm{P}$ but adding the commutator $[x, y] = x^{-1}y^{-1}xy$:
$\mathrm{Eq}(A_4; \cdot, e, ^{-1}, [.,.]) \in$ NP-c.

Possible fix

- investigate the 'hardest' polynomial extensions of **A**
- *Circuit satisfiability* CSat(**A**): terms are given by algebraic circuits

$[\ldots[[x_1, x_2], x_3] \ldots x_n]$ as $(\cdot, ^{-1})$-circuit
©Idziak, Krzaczkowski



The bad news

Complexity is not stable under taking quotients.
(Idziak, Kraczkowksi '17)

There is an algebra **A** with $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}) \in \mathrm{P}$, but $\mathrm{CSat}(\mathbf{A}/\theta) \in$ NP-c.

The bad news

Complexity is not stable under taking quotients. (Idziak, Kraczkowksi '17)

There is an algebra **A** with $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}) \in \mathrm{P}$, but $\mathrm{CSat}(\mathbf{A}/\theta) \in$ NP-c.

Possible fix

The bad news

Complexity is not stable under taking quotients.
(Idziak, Kraczkowksi '17)

There is an algebra **A** with $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}) \in \mathrm{P}$, but $\mathrm{CSat}(\mathbf{A}/\theta) \in$ NP-c.

Possible fix

- 'Intractability' = NP-completeness of some factor

The bad news

Complexity is not stable under taking quotients.
(Idziak, Kraczkowksi ’17)
There is an algebra **A** with $\theta \in \mathrm{Con}(\mathbf{A})$ such that CSat(**A**) $\in$ P, but CSat($\mathbf{A}/\theta$) $\in$ NP-c.

Possible fix

- 'Intractability' = NP-completeness of some factor
- Study only 'nice' classes of algebras

The bad news

Complexity is not stable under taking quotients.
(Idziak, Kraczkowksi '17)

There is an algebra $\mathbf{A}$ with $\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}) \in \mathrm{P}$, but $\mathrm{CSat}(\mathbf{A}/\theta) \in \mathrm{NP}$-c.

Possible fix

- 'Intractability' = NP-completeness of some factor
- Study only 'nice' classes of algebras

Question (Idziak, Kraczkowksi)

Is there an algebra $\mathbf{A}$ from a congruence permutable / modular variety, such that $\mathrm{CSat}(\mathbf{A}/\theta)$ is harder than $\mathrm{CSat}(\mathbf{A})$?

The good news

For certain classes of algebraic structures, much is known:

The good news

For certain classes of algebraic structures, much is known:

$\mathbf{A}$	$\mathrm{Eq}(\mathbf{A})$	extended $\mathrm{Eq}(\mathbf{A})$	$\mathrm{CSat}(\mathbf{A})$
Nilpotent Ring	P	P	P
Nilpotent Group	P	P	P
Non-nilpotent Ring	NP-c	NP-c	NP-c
Non-solvable Group	NP-c	NP-c	NP-c
Non-nilpotent Group	P, ?, NP-c	NP-c	NP-c

The good news

For certain classes of algebraic structures, much is known:

$\mathbf{A}$	Eq($\mathbf{A}$)	extended Eq($\mathbf{A}$)	CSat($\mathbf{A}$)
Nilpotent Ring	P	P	P
Nilpotent Group	P	P	P
Non-nilpotent Ring	NP-c	NP-c	NP-c
Non-solvable Group	NP-c	NP-c	NP-c
Non-nilpotent Group	P, ?, NP-c	NP-c	NP-c

Hypothesis

The good news

For certain classes of algebraic structures, much is known:

$\mathbf{A}$	Eq($\mathbf{A}$)	extended Eq($\mathbf{A}$)	CSat($\mathbf{A}$)
Nilpotent Ring	P	P	P
Nilpotent Group	P	P	P
Non-nilpotent Ring	NP-c	NP-c	NP-c
Non-solvable Group	NP-c	NP-c	NP-c
Non-nilpotent Group	P, ?, NP-c	NP-c	NP-c

Hypothesis

- Commutator theory might help in classifiying Eq($\mathbf{A}$)

The good news

For certain classes of algebraic structures, much is known:

$\mathbf{A}$	Eq($\mathbf{A}$)	extended Eq($\mathbf{A}$)	CSat($\mathbf{A}$)
Nilpotent Ring	P	P	P
Nilpotent Group	P	P	P
Non-nilpotent Ring	NP-c	NP-c	NP-c
Non-solvable Group	NP-c	NP-c	NP-c
Non-nilpotent Group	P, ?, NP-c	NP-c	NP-c

Hypothesis

- Commutator theory might help in classifiying Eq($\mathbf{A}$)
- In particular, if $\mathbf{A}$ belongs to a *congruence permutable*, or *congruence modular* variety

Nilpotent rings $\rightarrow$ supernilpotent algebras in CM varieties

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus CSat($\mathbf{A}$) $\in P$:

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow f - g = 0$.

Tractability for nilpotent rings

Theorem (Horváth '11)

For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow f - g = 0$.
- Test all tuples with at most d-many entries $\neq 0$.

Tractability for supernilpotent Mal'cev algebra

Theorem (Idziak, Krzaczkowski / MK '17)
For every nilpotent ring $\mathbf{A} = (A; +, \cdot, 0, 1)$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \dots, A) = \{p(a_1, a_2, \dots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \dots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow f - g = 0$.
- Test all tuples with at most d-many entries $\neq 0$.

Tractability for supernilpotent Mal'cev algebra

Theorem (Idziak, Krzaczkowski / MK '17)

For every supernilpotent $\mathbf{A}$ with Malcev term $m(x, y, z)$ and $0 \in A$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-nilpotency: Monomials $x_1 \cdot x_2 \cdots x_k \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow f - g = 0$.
- Test all tuples with at most d-many entries $\neq 0$.

Tractability for supernilpotent Mal'cev algebra

Theorem (Idziak, Krzaczkowski / MK '17)

For every supernilpotent $\mathbf{A}$ with Malcev term $m(x,y,z)$ and $0 \in A$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \ldots, A) = \{p(a_1, a_2, \ldots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-supernilpotency: commutator terms $t(x_1, \ldots, x_k)$ are trivial $\forall i : t(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_k) \approx 0 \to t(x_1, x_2, \cdots, x_k) \approx 0$
- every polynomial $p(x_1, \ldots, x_n)$ is equivalent to the sum of monomials of degree $< k$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow f - g = 0$.
- Test all tuples with at most d-many entries $\neq 0$.

Tractability for supernilpotent Mal'cev algebra

Theorem (Idziak, Krzaczkowski / MK '17)

For every supernilpotent $\mathbf{A}$ with Malcev term $m(x, y, z)$ and $0 \in A$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \dots, A) = \{p(a_1, a_2, \dots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-supernilpotency: commutator terms $t(x_1, \dots, x_k)$ are trivial $\forall i : t(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots x_k) \approx 0 \to t(x_1, x_2, \cdots, x_k) \approx 0$
- every polynomial $p(x_1, \dots, x_n)$ is equivalent to the "sum" of commutator terms of degree $< k$, where $x + y = m(x, 0, y)$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow f - g = 0$.
- Test all tuples with at most d-many entries $\neq 0$.

Tractability for supernilpotent Mal'cev algebra

Theorem (Idziak, Krzaczkowski / MK '17)

For every supernilpotent $\mathbf{A}$ with Malcev term $m(x,y,z)$ and $0 \in A$, there is a $d = d(\mathbf{A})$ such that the range of every polynomial

$$p(A, A, \dots, A) = \{p(a_1, a_2, \dots a_n) \mid \text{at most } d \text{ many } a_i \text{ are } \neq 0\}.$$

Proof idea

- k-supernilpotency: commutator terms $t(x_1, \dots, x_k)$ are trivial $\forall i : t(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots x_k) \approx 0 \rightarrow t(x_1, x_2, \cdots, x_k) \approx 0$
- every polynomial $p(x_1, \dots, x_n)$ is equivalent to the "sum" of commutator terms of degree $< k$, where $x + y = m(x, 0, y)$.
- this representation + a Ramsey argument gives us $d = d(|A|, k)$. $\square$

Thus $\mathrm{CSat}(\mathbf{A}) \in P$:

- It is enough to decide $f(\bar{x}) = 0$, as $f = g \leftrightarrow m(f, g, 0) = 0$
- Test all tuples with at most d-many entries $\neq 0$.

DL-like algebras

There is another class of tractable algebras:

DL-like algebras

There is another class of tractable algebras:

Example

Let $m(x, y, z)$ be the majority operation on $\{0, 1\}$.

DL-like algebras

There is another class of tractable algebras:

Example

Let $m(x, y, z)$ be the majority operation on $\{0, 1\}$. Every term is monotonous, hence

$$\begin{aligned} &\exists x_1, \dots, x_n f(x_1, \dots, x_n) = g(x_1, \dots, x_n) \\ \leftrightarrow\ & f(0, 0, \dots, 0) = g(0, 0, \dots, 0) \vee f(1, 1, \dots, 1) = g(1, 1, \dots, 1) \end{aligned}$$

Thus $\mathrm{Eq}(\{0, 1\}; m(x, y, z)) \in P$.

DL-like algebras

There is another class of tractable algebras:

Example

Let $m(x, y, z)$ be the majority operation on $\{0, 1\}$. Every term is monotonous, hence

$$\exists x_1, \dots, x_n f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$$
$$\leftrightarrow f(0, 0, \dots, 0) = g(0, 0, \dots, 0) \vee f(1, 1, \dots, 1) = g(1, 1, \dots, 1)$$

Thus $\mathrm{Eq}(\{0, 1\}; m(x, y, z)) \in P$.

Corollary

Let $\mathbf{A}$ be from a CM variety, and $\mathbf{A} = \mathbf{D} \times \mathbf{N}$, such that

DL-like algebras

There is another class of tractable algebras:

Example

Let $m(x, y, z)$ be the majority operation on $\{0, 1\}$. Every term is monotonous, hence

$$\begin{aligned} &\exists x_1, \dots, x_n f(x_1, \dots, x_n) = g(x_1, \dots, x_n) \\ \leftrightarrow\ & f(0, 0, \dots, 0) = g(0, 0, \dots, 0) \vee f(1, 1, \dots, 1) = g(1, 1, \dots, 1) \end{aligned}$$

Thus $\mathrm{Eq}(\{0, 1\}; m(x, y, z)) \in P$.

Corollary

Let $\mathbf{A}$ be from a CM variety, and $\mathbf{A} = \mathbf{D} \times \mathbf{N}$, such that

- $\mathbf{D}$ is *DL-like*: subdirect product of algebras polynomially equivalent to $(\{0, 1\}; \wedge, \vee)$,

DL-like algebras

There is another class of tractable algebras:

Example

Let $m(x, y, z)$ be the majority operation on $\{0, 1\}$. Every term is monotonous, hence

$$\begin{aligned} &\exists x_1, \ldots, x_n f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \\ \leftrightarrow\ & f(0, 0, \ldots, 0) = g(0, 0, \ldots, 0) \vee f(1, 1, \ldots, 1) = g(1, 1, \ldots, 1) \end{aligned}$$

Thus $\mathrm{Eq}(\{0, 1\}; m(x, y, z)) \in P$.

Corollary

Let $\mathbf{A}$ be from a CM variety, and $\mathbf{A} = \mathbf{D} \times \mathbf{N}$, such that

- $\mathbf{D}$ is *DL-like*: subdirect product of algebras polynomially equivalent to $(\{0, 1\}; \wedge, \vee)$,
- $\mathbf{N}$ is supernilpotent.

DL-like algebras

There is another class of tractable algebras:

Example

Let $m(x,y,z)$ be the majority operation on $\{0,1\}$. Every term is monotonous, hence

$$\begin{aligned} &\exists x_1,\dots,x_n f(x_1,\dots,x_n)=g(x_1,\dots,x_n) \\ \leftrightarrow\ & f(0,0,\dots,0)=g(0,0,\dots,0)\vee f(1,1,\dots,1)=g(1,1,\dots,1) \end{aligned}$$

Thus $\mathrm{Eq}(\{0,1\};m(x,y,z))\in P$.

Corollary

Let $\mathbf{A}$ be from a CM variety, and $\mathbf{A}=\mathbf{D}\times\mathbf{N}$, such that

- $\mathbf{D}$ is *DL-like*: subdirect product of algebras polynomially equivalent to $(\{0,1\};\wedge,\vee)$,
- $\mathbf{N}$ is supernilpotent.

Then $\mathrm{CSat}(\mathbf{A})\in$P.

Intractability in the non-nilpotent case

What about hardness result?

Intractability in the non-nilpotent case

What about hardness result?

Theorem (Idziak, Krzaczkowski '17)

For every algebra $\mathbf{A}$ in a CM variety, that is not the direct product of a nilpotent algebra and a DL-like algebra $\exists\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}/\theta) \in$ NP-c.

Intractability in the non-nilpotent case

What about hardness result?

Theorem (Idziak, Krzaczkowski ’17)

For every algebra **A** in a CM variety, that is not the direct product of a nilpotent algebra and a DL-like algebra $\exists\theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}/\theta) \in$ NP-c.

Summary

Let **A** be an algebra from a congruence modular variety. Then

A	Eq(**A**)	CSat(**A**)	CSat(**A**/θ)
DL-l. $\times$ supernilpotent	P	P	P
not (DL-l. $\times$ nilpotent)	P,?,NP-c	?, NP-c	$\exists\theta$: NP-c

Intractability in the non-nilpotent case

What about hardness result?

Theorem (Idziak, Krzaczkowski '17)

For every algebra **A** in a CM variety, that is not the direct product of a nilpotent algebra and a DL-like algebra $\exists \theta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{CSat}(\mathbf{A}/\theta) \in$ NP-c.

Summary

Let **A** be an algebra from a congruence modular variety. Then

A	Eq(**A**)	CSat(**A**)	CSat(**A**/θ)
DL-l. $\times$ supernilpotent	P	P	P
nilpotent, not supernilpotent	?	?	$\leq_p$ CSat(**A**)
not (DL-l. $\times$ nilpotent)	P,?,NP-c	?, NP-c	$\exists \theta$: NP-c

Work in progress...

Nilpotent, non-supernilpotent algebras

Example

Let p, q be distinct primes. The group expansion $(\mathbb{Z}_p \times \mathbb{Z}_q; +, f(x))$ with

$$f((x_1, x_2)) = \begin{cases} (0,1) \text{ if } x_1 = 0, \\ (0,0) \text{ else,} \end{cases}$$

is nilpotent but not supernilpotent (Aichinger + Mayr '07).

Nilpotent, non-supernilpotent algebras

Example

Let p, q be distinct primes. The group expansion $(\mathbb{Z}_p \times \mathbb{Z}_q; +, f(x))$ with

$$f((x_1, x_2)) = \begin{cases} (0,1) \text{ if } x_1 = 0, \\ (0,0) \text{ else,} \end{cases}$$

is nilpotent but not supernilpotent (Aichinger + Mayr '07).

- We can show that $\mathrm{CSat}(\mathbb{Z}_p \times \mathbb{Z}_2; +, f(x)) \in \mathrm{P}$.

Nilpotent, non-supernilpotent algebras

Example

Let p, q be distinct primes. The group expansion $(\mathbb{Z}_p \times \mathbb{Z}_q; +, f(x))$ with

$$f((x_1, x_2)) = \begin{cases} (0,1) \text{ if } x_1 = 0, \\ (0,0) \text{ else,} \end{cases}$$

is nilpotent but not supernilpotent (Aichinger + Mayr '07).

- We can show that $\mathrm{CSat}(\mathbb{Z}_p \times \mathbb{Z}_2; +, f(x)) \in \mathrm{P}$.
- However, in every arity n there is a polynomial $p_n(x_1, \ldots, x_n)$, such that $\mathrm{Eq}(\mathbb{Z}_p \times \mathbb{Z}_q; +, f, (p_n)_{n \in \mathbb{N}}) \in$ NP-c!

Nilpotent, non-supernilpotent algebras

Example

Let p, q be distinct primes. The group expansion $(\mathbb{Z}_p \times \mathbb{Z}_q; +, f(x))$ with

$$f((x_1, x_2)) = \begin{cases} (0,1) \text{ if } x_1 = 0, \\ (0,0) \text{ else,} \end{cases}$$

is nilpotent but not supernilpotent (Aichinger + Mayr '07).

- We can show that $\mathrm{CSat}(\mathbb{Z}_p \times \mathbb{Z}_2; +, f(x)) \in \mathrm{P}$.
- However, in every arity n there is a polynomial $p_n(x_1, \ldots, x_n)$, such that $\mathrm{Eq}(\mathbb{Z}_p \times \mathbb{Z}_q; +, f, (p_n)_{n\in\mathbb{N}}) \in$ NP-c!

Remark

This is a phenomenon that might appear in other nilpotent, non-supernilpotent algebras $\mathbf{A}$, as for every arity n there is a non-trivial commutator term $p_n(x_1, \ldots, x_n)$.

Thank you!