Difference clonoids and their applications

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Central congruences

$\underline{A}$ ... algebra
$Clo(\underline{A})$ ... term clone

$\alpha \in Con(\underline{A})$ is central $:\Leftrightarrow$
$\forall t \in Clo(\underline{A})\ \forall x \underset{\alpha}{\sim} y, \forall \bar{u}, \bar{v} \in A^n$

$$\left\{ \begin{array}{ll} & t(x,\bar{u}) = t(x,\bar{v}) \\ \Leftrightarrow & t(y,\bar{u}) = t(y,\bar{v}) \end{array} \right.$$

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Example:
$\underline{G} = (G,\cdot,1,^{-1})$
group

$\alpha \in \mathrm{Con}(\underline{G})$ is central $\Longleftrightarrow$
$[1]_\alpha \trianglelefteq G$ is central subgroup.

Abelianness

$$\begin{cases} \bullet\ 1_{\underline{A}} = A \times A \text{ is central} \\ \qquad\qquad \text{„}\underline{A} \text{ is Abelian“} \\ \bullet\ \underline{A} \text{ satisfies } (*) \end{cases}$$

$\Rightarrow$ $\underline{A}$ is affine

$$\boxed{\mathrm{Clo}(\underline{A}, (a)_{a \in A}) = \{\bar{x} \mapsto \sum_{i=1}^{n} r_i x_i + a\} \text{ in some module}}$$

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•) $\underline{A}$ has a (weak) difference term.

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What can be said about central congruences in general?

Central extensions / wreath products

$$\begin{cases} \underline{A} = (A, (f^{\underline{A}})_{f \in \mathcal{F}}) \text{ satisfies } (*) \\ \alpha \in \mathrm{Con}\,\underline{A} \text{ central} \quad \underline{U} := \underline{A}/\alpha \end{cases}$$

$\Rightarrow \exists\, \underline{L}$ affine:

central extension

$$\underline{A} \cong \underline{L} \otimes \underline{U}$$

Domain $L \times U$

$$f^{\underline{A}}\left(\begin{pmatrix} l_1 \\ u_1 \end{pmatrix} \cdots \begin{pmatrix} l_n \\ u_n \end{pmatrix}\right) = \begin{pmatrix} f^{\underline{L}}(l_1, \ldots, l_n) + \hat{f}(u_1, \ldots, u_n) \\ f^{\underline{U}}(u_1, \ldots, u_n) \end{pmatrix}$$

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$\hat{f} : U^n \to L \qquad T = (\hat{f})_{f \in \mathcal{F}}$

$0 \in L$ ...NE f^+

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$0 \in L$ ... NE f^+ wlog. $f^{L}(0 \dots 0) = 0$

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A clone homomorphism

$\underline{A} = \underline{L} \otimes \underline{U}$

$$\boxed{\begin{aligned} f : \mathrm{Clo}(\underline{A}) &\longrightarrow \mathrm{Clo}(\underline{L} \times \underline{U}) \\ t^{\underline{A}} &\longmapsto t^{\underline{L} \times \underline{U}} \end{aligned}}$$

is a clone homomorphism

- $f(\pi_i^n) = \pi_i^n$
- $f(f \circ (g_1 \dots g_n)) = f(f) \circ (f(g_1) \dots f(g_n))$

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Question:

- when is $Clo(\underline{L} \times \underline{U}) \leq Clo(\underline{A})$?
- when $\exists \eta : Clo(\underline{L} \times \underline{U}) \to Clo(\underline{A})$ retraction of f?

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Motivation

Lift results about $\underline{L}\times\underline{U}$ and $(\hat{f})_{f\in F}$ to $\underline{A}$.

Problem: Not always possible, e.g. $\underline{A} = \underline{D}_4$!

What else can we do?

Difference clonoids (Mayr)

Let $t^{\underline{A}}, s^{\underline{A}} \in \mathrm{Clo}(\underline{A})$: $\xi(t^{\underline{A}}) = \xi(s^{\underline{A}})$

$$t^{\underline{A}} \qquad\qquad\qquad s^{\underline{A}}$$

$$\begin{pmatrix} t^{\underline{L}}(\bar{e}) + \hat{f}(\bar{u}) \\ t^{\underline{U}}(\bar{u}) \end{pmatrix} \qquad \begin{pmatrix} t^{\underline{L}}(\bar{e}) + \hat{s}(\bar{u}) \\ t^{\underline{U}}(\bar{u}) \end{pmatrix}$$

Def The <u>difference</u> $t^{\underline{A}} - s^{\underline{A}}$ is the map $\hat{r}: U^n \to L$

$$\hat{r}(\bar{u}) = \hat{f}(\bar{u}) - \hat{s}(\bar{u})$$

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$$\boxed{\mathrm{Diff}(\underline{A}) := \{\hat{r} : U^n \to L \mid \hat{r} = t^{\underline{A}} - s^{\underline{A}}\}}$$

difference clonoid of $\underline{L} \otimes \underline{U}$

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- $\mathrm{Diff}(\underline{A})$ describes the kernel of $\xi: \mathrm{Clo}(\underline{A}) \to \mathrm{Clo}(\underline{L} \times \underline{U})$
- $\mathrm{Diff}(\underline{A})$ is a clonoid from $\underline{U}$ to $(\underline{L}, 0)$

$$\mathrm{Clo}(\underline{L}, 0) \circ \mathrm{Diff}(\underline{A}) \circ \mathrm{Clo}(\underline{U}) = \mathrm{Diff}(\underline{A})$$

Why is it a clonoid?

•) If $\hat{r} = t^{\underline{A}} - s^{\underline{A}}$

then $\hat{r} \circ (g_1^{\underline{U}} \ldots g_n^{\underline{U}}) = t^{\underline{A}} \circ (g_1^{\underline{A}} \ldots g_n^{\underline{A}}) - s^{\underline{A}} \circ (g_1^{\underline{A}} \ldots g_n^{\underline{A}})$

$\Rightarrow$ closed under $Clo(\underline{U})$

•) If $d(x,y,z)$ difference term / Mal'tsev term of $\underline{A}$

$\hat{r} = t^{\underline{A}} - s^{\underline{A}}$

$$d(f, s, t) \cdots \begin{pmatrix} f^{\underline{L}}(\bar{e}) + \hat{f}(\bar{u}) + \hat{r}(\bar{u}) \\ f^{\underline{U}}(\bar{u}) \end{pmatrix} = \text{,,} \hat{f}^{\underline{A}} + \hat{r}\text{''}$$

$\Rightarrow$ closed under $(L, +)$

The Subpower Membership Problem (SMP)

SMP($\underline{A}$)

Input: $\bar{a}_1, \dots \bar{a}_n, \bar{b} \in A^k$

Question: $\exists t \in Clo(\underline{A})$: $t(\bar{a}_1, \dots \bar{a}_n) = \bar{b}$?

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→ makes also sense for $(\underline{U}, \underline{L})$-clonoids $\mathcal{C}$:

SMP($\mathcal{C}$):

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$\underline{A} = \underline{L} \otimes \underline{U}$, Mal'tsev, finite

Theorem [MK '24]

- If $\mathrm{Clo}(\underline{L} \times \underline{U}) \subseteq \mathrm{Clo}(\underline{A})$:

$$\mathrm{SMP}(\underline{A}) \sim_{p\text{-time}} \mathrm{SMP}(\underline{L} \times \underline{U}) \wedge \mathrm{SMP}(\mathrm{Diff}(\underline{A}))$$

- If $\underline{U}$ is supernilpotent

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$\boxed{\begin{array}{l} \text{•) If } \underline{U} \text{ is supernilpotent} \\ \qquad SMP(\underline{A}) \sim_{p\text{-time}} SMP(Diff(\underline{A})) \end{array}}$

If $\underline{U} = \mathbb{Z}_p$ prime $p \nmid |L|$

[Fioravanti '20] + ε [MK '24]

$\Downarrow$

$SMP(\underline{A}) \in P$.

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[Mayr, Wynne '23]

If $\begin{cases} \underline{U}, \underline{L} \text{ affine, coprime} \\ Con(\underline{U}) \text{ distributive} \\ \mathcal{C} \text{ is } (\underline{U}, \underline{L})\text{-clonoid} \end{cases}$

$\Rightarrow$ $\mathcal{C}$ is fin. generated & "nice".

ε [TO DO]

$SMP(\underline{A}) \in P$

Finite equational basis

$\mathcal{C}$ ... $(\underline{U}, \underline{L})$-clonoid is finitely based :$\Leftrightarrow$

- $\cdot)$ Generated by $T = \{\hat{f}_1, \dots, \hat{f}_n\}$ $\hat{f}_i : U^{k_i} \to L$
- $\cdot)$ $\mathrm{Id}(\underline{U}, T, \underline{L})$ is finitely based

For $\underline{A} = \underline{L} \otimes \underline{U}$, goal:

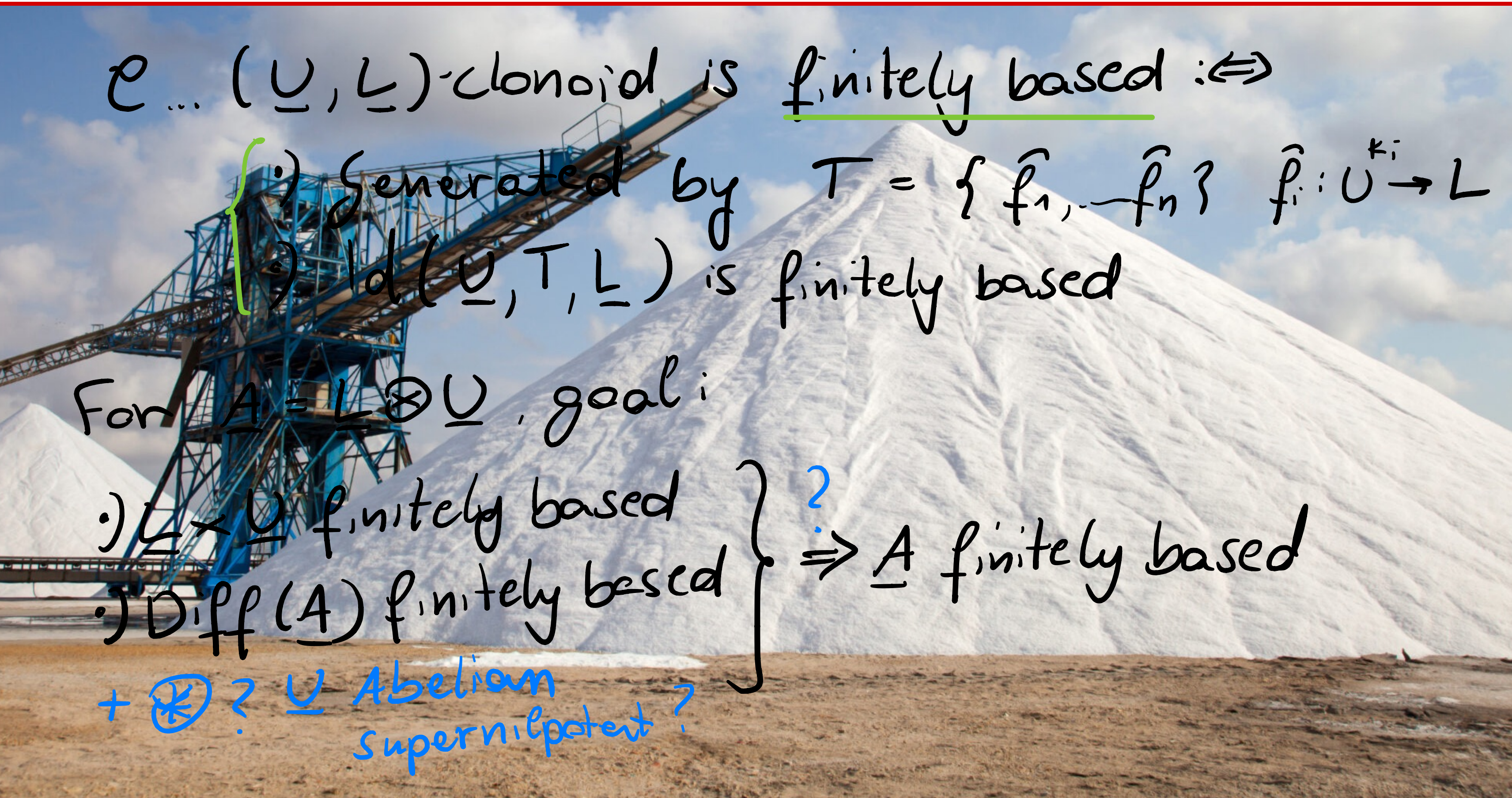
$\cdot)$ $\underline{L} \times \underline{U}$ finitely based

$\cdot)$ $\mathrm{Diff}(\underline{A})$ finitely based

$\overset{?}{\Rightarrow}$ $\underline{A}$ finitely based

+ $(*)$? $\underline{U}$ Abelian supernilpotent ?

Finite equational basis



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+ ⊛ ? $\underline{U}$ Abelian supernilpotent ?

Thank you for your attention!

Any questions?