EXERCISE 4 - DELTA METHOD FOR POINTWISE CI

Kaplan-Meier estimator $\hat{S}(t) = \prod_{\{j: t_j \le t\}} \left[1 - \frac{\Delta \bar{N}(t_j)}{\bar{Y}(t_j)}\right]$

Theorem 4.8 gives us $\sqrt{n}\left[\hat{S}(t) - S(t)\right] \xrightarrow{w} S(t) \cdot W(\sigma(t))$ on $D[0,\tau]$ where $\sigma(t) = \int_0^t \frac{\lambda(s)}{\pi(s)}\,ds$

$\pi(s) = P(Y_1(s) = 1) > 0$ on $[0,\tau]$

This convergence of processes implies convergence of any finite-dimensional distribution, therefore,

for any $t \in [0,\tau]$: $\sqrt{n}\left[\hat{S}(t) - S(t)\right] \xrightarrow[n\to\infty]{d} N(0, S(t)\sigma(t))$

estimate by $\hat{S}^2(t)$ Greenwood formula: $\hat{\sigma}(t) = n \cdot \sum_{\{j: t_j \le t\}} \frac{\Delta \bar{N}(t_j)}{\bar{Y}(t_j)\left(\bar{Y}(t_j) - \Delta \bar{N}(t_j)\right)}$

Cramér-Slutsky: $\sqrt{n}\,\frac{\hat{S}(t) - S(t)}{\hat{S}(t)\sqrt{\hat{\sigma}(t)}} = \underbrace{\sqrt{n}\,\frac{\hat{S}(t) - S(t)}{S(t)\sqrt{\sigma(t)}}}_{\xrightarrow[n\to\infty]{d} N(0,1)} \cdot \underbrace{\frac{S(t)\cdot\sqrt{\sigma(t)}}{\hat{S}(t)\cdot\sqrt{\hat{\sigma}(t)}}}_{\xrightarrow[n\to\infty]{P} 1} \xrightarrow[n\to\infty]{d} N(0,1)$

Δ-theorem for functions $g: [0,1] \to \mathbb{R}$ $\quad \sqrt{n}\,\frac{g(\hat{S}(t)) - g(S(t))}{\hat{S}(t)\sqrt{\hat{\sigma}(t)}} \xrightarrow[n\to\infty]{d} N\left(0, \left[g'(S(t))\right]^2\right)$

Again by Cramér-Slutsky: $\sqrt{n}\,\frac{g(\hat{S}(t)) - g(S(t))}{\hat{S}(t)\,|g'(\hat{S}(t))| \cdot \sqrt{\hat{\sigma}(t)}} \xrightarrow[n\to\infty]{d} N(0,1)$

$$1-\alpha \underset{n\to\infty}{\leftarrow} P\left(|g(\hat{S}(t)) - g(S(t))| \le u_{1-\frac{\alpha}{2}} \cdot \hat{S}(t) \cdot |g'(\hat{S}(t))| \cdot \sqrt{\frac{\hat{\sigma}(t)}{n}} \right) =$$

$$= P\left(g(\hat{S}(t)) - u_{1-\frac{\alpha}{2}} |g'(\hat{S}(t))| \hat{S}(t) \sqrt{\frac{\hat{\sigma}(t)}{n}} \le g(S(t)) \le g(\hat{S}(t)) + u_{1-\frac{\alpha}{2}} \hat{S}(t) |g'(\hat{S}(t))| \sqrt{\frac{\hat{\sigma}(t)}{n}} \right)$$

$$\underset{g \text{ increasing}}{=} P\left(g^{-1}\left[g(\hat{S}(t)) - u_{1-\frac{\alpha}{2}} \hat{S}(t) \sqrt{\frac{\hat{\sigma}(t)}{n}} |g'(\hat{S}(t))| \right] \le S(t) \le g^{-1}\left[g(\hat{S}(t)) + u_{1-\frac{\alpha}{2}} \hat{S}(t) |g'(\hat{S}(t))| \sqrt{\frac{\hat{\sigma}(t)}{n}} \right] \right)$$

Choices of g: • $g(y) = \log y$, $g'(y) = \frac{1}{y}$, $g^{-1}(y) = \exp\{y\}$

$$\exp\left\{ \log(\hat{S}(t)) \pm u_{1-\frac{\alpha}{2}} \frac{\hat{S}(t)}{\hat{S}(t)} \cdot \sqrt{\frac{\hat{\sigma}(t)}{n}} \right\} = \hat{S}(t) \cdot e^{\pm u_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\sigma}(t)}{n}}}$$

• $g(y) = \log(-\log(y))$, $g'(y) = \frac{1}{-\log(y)} \cdot \frac{(-1)}{y} = \frac{-1}{-y\log y}$ (decreasing), $g^{-1}(y) = \exp\{-\exp\{y\}\}$

$$\exp\left\{ -\exp\left\{ \log(-\log(\hat{S}(t))) \mp u_{1-\frac{\alpha}{2}} \cdot \frac{\hat{S}(t)}{-\hat{S}(t)\log(\hat{S}(t))} \cdot \sqrt{\frac{\hat{\sigma}(t)}{n}} \right\}\right\} = \left[\hat{S}(t)\right]^{\exp\left\{ \pm \frac{u_{1-\frac{\alpha}{2}} \sqrt{\hat{\sigma}(t)}}{\log \hat{S}(t) \sqrt{n}} \right\}}$$

• $g(y) = \text{logit}(y) = \log \frac{y}{1-y}$, $g'(y) = \frac{1-y}{y} \cdot \frac{(1-y)+y}{(1-y)^2} = \frac{1}{y(1-y)}$ $\quad g^{-1}(y) = \frac{\exp\{y\}}{1+\exp\{y\}} = \frac{1}{1+\exp\{-y\}}$

$$\text{logit}^{-1}\left(\text{logit}(\hat{S}(t)) \pm u_{1-\frac{\alpha}{2}} \hat{S}(t) \sqrt{\frac{\hat{\sigma}(t)}{n}} \cdot \frac{1}{\hat{S}(t)(1-\hat{S}(t))} \right) = \text{logit}^{-1}\left(\text{logit}(\hat{S}(t)) \pm u_{1-\frac{\alpha}{2}} \frac{\sqrt{\hat{\sigma}(t)}}{(1-\hat{S}(t))\sqrt{n}} \right)$$