Three factors $\quad X, Z, V$

$\pi_{ijk} = P(X=i;\ Z=j;\ V=k)$

$m_{ijk} = n \cdot \pi_{ijk}$ (or $(\mathbb{E}n)\,\pi_{ijk}$) : expected count

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Saturated model

$$\log(m_{ijk}) = \beta_0 + \beta_i^X + \beta_j^Z + \beta_k^V + \beta_{ij}^{XZ} + \beta_{ik}^{XV} + \beta_{jk}^{ZV} + \beta_{ijk}^{XZV}$$

identifiability ($\equiv$ dummy variables)

$\beta_1^X = 0, \quad \beta_1^Z = 0, \quad \beta_1^V = 0$

$\beta_{1j}^{XZ} = 0, \quad \beta_{i1}^{XZ} = 0, \quad \ldots$

$\beta_{1jk}^{XZV} = 0, \quad \ldots$

As with two factors:

$P(X=i_1, Z=*, V=$ [illegible] $)$ / $P(X=i_2, Z=*, V=$ [illegible] $)$ = $P(X=i_1 \mid Z=*, V=$ [illegible] $)$ / $P(X=i_2 \mid Z=*, V=$ [illegible] $)$

$$\rightarrow \exp(\beta_i^X) = \frac{P(X=i \mid Z=1, V=1)}{P(X=1 \mid Z=1, V=1)} =: \text{odds}_X(i,1 \mid Z=1, V=1)$$

two-way interactions

$$\exp(\beta_i^X + \beta_{ij}^{XZ}) = \frac{P(X=i \mid Z=j, V=1)}{P(X=1 \mid Z=j, V=1)}$$

$$=: \text{odds}_X(i,1 \mid Z=j, V=1)$$

$$\Rightarrow \exp(\beta_{ij}^{XZ}) = \frac{\text{odds}_X(i,1 \mid Z=j, V=1)}{\text{odds}_X(i,1 \mid Z=1, V=1)}$$

$$=: OR_X(i,1 \mid Z: j \leftrightarrow 1, V=1)$$

$$= OR_Z(j,1 \mid X: i \leftrightarrow 1, V=1)$$

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three-way interactions

$$\exp(\beta_{ijk}^{XZV}) = \frac{OR_X(i,1 \mid Z: j \leftrightarrow 1, V=k)}{OR_X(i,1 \mid Z: j \leftrightarrow 1, V=1)}$$

$$= \frac{OR_Z(j,1 \mid X: i \leftrightarrow 1, V=k)}{OR_Z(j,1 \mid X: i \leftrightarrow 1, V=1)}$$

$$= \ldots\ldots$$

Association structures $X \leftrightarrow Z$ under presence of V

$$OR_X(i1, i2 \mid Z: j1 \leftrightarrow j2, V=k) =$$

$$= \frac{odds_X(i1, i2 \mid Z=j1, V=k)}{odds_X(i1, i2 \mid Z=j2, V=k)} \qquad (*)$$

$$OR_Z(j1, j2 \mid X: i1 \leftrightarrow i2, V=k) =$$

$$= \frac{odds_Z(j1, j2 \mid X=i1, V=k)}{odds_Z(j1, j2 \mid X=i2, V=k)}$$

$\equiv$ association $X \leftrightarrow Z$ generally depends on $V (=k)$

Saturated model

$$X + Z + V + X{:}Z + X{:}V + Z{:}V + X{:}Z{:}V$$

$$\equiv V + (X + X(V)) + (Z + Z(V))$$

$$+ (X{:}Z + X{:}Z(V))$$

$\equiv$ given V, it is two-way interaction model for $X \leftrightarrow Z$ association

assoc. structure $X \leftrightarrow Z$ depends on V

No three-way interaction

$X + Z + V + X:Z + X:V + Z:V$

$\rightarrow$ OR's $\forall$ do not depend on $(V=k)$

$OR_X(i1, i2 \,/\, Z: j1 \leftrightarrow j2, V=1) = \dots = OR_X(i1, i2 \,/\, Z: j1 \leftrightarrow j2, V=k)$

$OR_Z(j1, j2 \,/\, X: i1 \leftrightarrow i2, V=1) = \dots = OR_Z(j1, j2 \,/\, X: i1 \leftrightarrow i2, V=k)$

$= V + (X + X(V)) + (Z + Z(V)) + \underbrace{X:Z}_{\equiv \text{ no } V \text{ here}}$

- given V: X, Z dependent
- $X \leftrightarrow Z$ association does not depend on V

$\equiv$ homogeneous associations between X and Z given V

X:Z interaction in the model (but not X:Z:V, not all two-way inter.)

$X + Z + V + X{:}Z$

$X + Z + V + X{:}Z + X{:}V$

$X + Z + V + X{:}Z : Z{:}V$

→ again homogeneous association

- $X + Z + V + X{:}Z$

  $\equiv V + (X + Z + X{:}Z)$

  $\equiv (X, Z) \perp\!\!\!\perp V$ (stronger than)

$X:Z$ not in the model

$$X+Z+V+X:V+Z:V \equiv V+(X+X(V))+(Z+Z(V))$$

$$X+Z+V+X:V \quad V+(X+X(V))+Z$$

$$X+Z+V+Z:V \quad V+X+(Z+Z(V))$$

$$X+Z+V \quad V+X+V$$

$\equiv X \perp\!\!\!\perp Z$ given V

SUMMARY

- **$X:Z:V$ in model**

  → conditional association $X \leftrightarrow Z$ depends on V

- **$X:Z:V$ not in model but $X:Z$ included**

  → homogeneous conditional association between X and Z given V

- **$X:Z$ not in model**

  → conditional independence X and Z given V