13.2 Two-way classification

- two categorical covariates $(p=2)$

  $Z \in \{1, .., G\} = \mathcal{Z}$

  $W \in \{1, .., H\} = \mathcal{W}$

  → two "factors" that may influence the outcome Y

  = division of "population" into $G \cdot H$ subpopulations identified by (Z, W)

  → two-way classification (dvojné třídění)

e.g. Z = type of machine (A, B, C)

W = temperature used for production $(20°C, 25°C, 30°C)$

OR Z = gender (M, F)

W = nationality $(CZ/SK/other)$

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response expected values:

$\mathbb{E}(Y|Z=g, W=h) = m(g,h) =: m_{g,h}$, $g = 1, .., G$

$h = 1, .., H$

- by LM, we may somehow parameterize this set of $G \cdot H$ two-way classified group means

$n_{gh} := \#(Z=g, W=h)$ in data

$$n = \sum_{g=1}^{G} \sum_{h=1}^{H} n_{g,h}$$

ASSUMPTION: $P(n_{gh} > 0) = 1 \quad \forall g, h$

(each (Z, W) combination represented a.s. in data) without loss of generality 14

$\to$ data sorted by Z and W values

$\to$ triple subscript used

$$
(Z,W) = \begin{pmatrix}
1 & 1 \\
\vdots & \vdots \\
1 & 1 \\
\hline
\vdots & \vdots \\
\hline
G & 1 \\
\vdots & \vdots \\
G & 1 \\
\hline
| & | \\
\hline
1 & H \\
\vdots & \vdots \\
1 & H \\
\hline
\vdots & \vdots \\
\hline
G & H \\
\vdots & \vdots \\
G & H
\end{pmatrix}
\qquad
\begin{pmatrix}
y_{1,1,1} \\
\vdots \\
y_{1,1,n_{11}} \\
\hline
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
=
\begin{pmatrix}
Y_{11} \\
\vdots \\
Y_{G,1} \\
\hline
\vdots \\
\hline
Y_{1,H} \\
\vdots \\
Y_{G,H}
\end{pmatrix}
= Y
$$

~~Response~~ Two-way classified group means 15

$$m = (m_{11}, \dots, m_{G,1}, \dots, m_{1,H}, \dots, m_{G,H})^T$$

(obvious) notation: 16

$$n_{g\cdot} = \sum_{h=1}^{H} n_{g,h}, \quad g = 1, .., G$$

$$n_{\cdot h} = \sum_{g=1}^{G} n_{g,h}, \quad h = 1, .., H$$

$$\overline{m} := \frac{1}{G \cdot H} \sum_{g=1}^{G} \sum_{h=1}^{H} m_{g,h}$$

$$\overline{m_{g\cdot}} := \frac{1}{H} \sum_{h=1}^{H} m_{g,h} \qquad , g = 1, .., G$$

$$\overline{m_{\cdot h}} := \frac{1}{G} \sum_{g=1}^{G} m_{g,h} \qquad , h = 1, .., H$$

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Remark

If $P(Z=g, W=h) = \frac{1}{G \cdot H}$ then

$$\bar{m} = \frac{1}{G \cdot H} \sum_g \sum_h \underbrace{\mathbb{E}(Y \mid Z=g, W=h)}_{m_{g,h}} = \mathbb{E} Y$$

$$\bar{m}_{g\cdot} = \frac{1}{H} \sum_{h=1}^{H} \mathbb{E}(Y \mid Z=g, W=h) = \mathbb{E}(Y \mid Z=g)$$

$$\bar{m}_{\cdot h} = \frac{1}{G} \sum_{g=1}^{G} \mathbb{E}(Y \mid Z=g, W=h) = \mathbb{E}(Y \mid W=h)$$

BUT NOT IN GENERAL!

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Everything in matrices

→ one more line to two-way classification

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By a linear model, we may "model"

$$\mathbb{E}(Y / Z, W) = \begin{pmatrix} m_{1,1} \mathbb{1}_{n_{11}} \\ \vdots \\ m_{G,1} \mathbb{1}_{n_{G,1}} \\ \hline \vdots \\ \hline m_{1,H} \mathbb{1}_{n_{1,H}} \\ \vdots \\ m_{G,H} \mathbb{1}_{n_{G,H}} \end{pmatrix}$$

ASSUMPTION (if LM used):

$$\operatorname{var}(Y / Z, W) = \sigma^2 I_n$$

(homoscedasticity)

13.2.1 Parameters of interest

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The mean of the group means

$$\overline{m} = \frac{1}{G \cdot H} \sum_{g=1}^{G} \sum_{h=1}^{H} m_{g,h} = \mathbb{E}\,Y \quad \left(= \sum_g \sum_h \mathbb{E}(Y/Z=g, W=h) \cdot P(Z=g, W=h)\right)$$

↑ if $P(Z=g, W=h) = \frac{1}{G \cdot H}$

+ comment on the slide

The means of the means by the first or the second factor

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$$\overline{m_{g\bullet}} = \frac{1}{H} \sum_{h=1}^{H} m_{g,h} = \mathbb{E}(Y/Z=g) \quad \left(= \sum_h \mathbb{E}(Y/Z=g, W=h) \cdot P(W=h)\right)$$

↑ if $P(W=h) = \frac{1}{H}$

$$\overline{m_{\bullet h}} = \frac{1}{G} \sum_{g=1}^{G} m_{g,h} = \mathbb{E}(Y/W=h) \quad \left(= \sum_g \mathbb{E}(Y/Z=g, W=h) \cdot P(Z=g)\right)$$

↑ if $P(Z=g) = \frac{1}{G}$

+ comment on the slide

Differences between the means of the means by the first or the second factor

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$$\theta_{g_1, g_2 \bullet} := \overline{m_{g_1 \bullet}} - \overline{m_{g_2 \bullet}}$$

$$\theta_{\bullet h_1, h_2} := \overline{m_{\bullet h_1}} - \overline{m_{\bullet h_2}}$$

+ comment on the slide

Def 13.2 Factor main effects in two-way classification

Short: $\eta_g^Z := \overline{m}_{g\circ} - \overline{m}$; $g = 1, .., G$

$\eta_h^W := \overline{m}_{\circ h} - \overline{m}$, $h = 1, .., H$.

Remark:

$$\vartheta_{g_1, g_2 \circ} = \overline{m}_{g_1 \circ} - \overline{m}_{g_2 \circ} = \eta_{g_1}^Z - \eta_{g_2}^Z$$

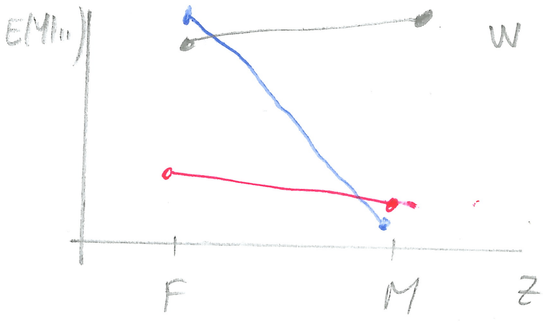
$$\vartheta_{\circ h_1, h_2} = \overline{m}_{\circ h_1} - \overline{m}_{\circ h_2} = \eta_{h_1}^W - \eta_{h_2}^W$$

13.2.2 Two-way ANOVA models

- different linear models
  $\equiv$ different structure imposed on the two-way classified group means

Interaction model $\quad M_{ZW}: \sim Z + W + Z:W$

- no structure of the group means
- LM = just some parameteriz. of $G \cdot H$ group means



Full-rank param:

$$m_{g,h} = \underbrace{\beta_0}_{1} + \underbrace{c_g^T}_{G-1} \beta^Z + \underbrace{d_h^T}_{H-1} \beta^W + \underbrace{(d_h^T \otimes c_g^T)}_{(G-1)\cdot(H-1)} \beta^{ZW}$$

$$\alpha_0 \quad + \quad \alpha_g^Z \quad + \quad \alpha_h^W \quad + \quad \alpha_{g,h}^{ZW}$$

$$\underset{G-1}{\mathbb{C} = \begin{pmatrix} c_1^T \\ \vdots \\ c_G^T \end{pmatrix}}, \quad \underset{H-1}{\mathbb{D} = \begin{pmatrix} d_1^T \\ \vdots \\ d_H^T \end{pmatrix}} \equiv \text{(pseudo)contrast matrices to parameterize } Z \text{ and } W$$

just for better interpretation:

$\alpha_0 := \beta_0$

$\alpha_g^Z := c_g^T \beta^Z$

$\alpha_h^W := d_h^T \beta^W$

$\alpha_{g,h}^{ZW} := (d_h^T \otimes c_g^T) \beta^{ZW}$

rank $= G \cdot H$ if $n_{g,h} > 0 \ \forall g,h$

plots 24-25

It was derived in Section 5.4:

With sum contrasts $\mathbb{C} = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 1 \\ \hline -1 & \cdots & -1 \end{pmatrix}$ $\mathbb{D} = \begin{pmatrix} 1 & \cdots & 0 \\ & \cdots & 1 \\ \hline -1 & \cdots & -1 \end{pmatrix}$

$$\alpha_0 = \beta_0 = \overline{m}$$

$$\left.\begin{array}{ll} \alpha_g^Z = \beta_g^Z = \overline{m}_{g\cdot} - \overline{m} = \eta_g^Z & , \; g = 1, \dots, G-1 \\ \alpha_G^Z = -\sum_{g=1}^{G-1} \beta_g^Z = \overline{m}_{G\cdot} - \overline{m} = \eta_G^Z & \\ \alpha_h^W = \beta_h^W = \overline{m}_{\cdot h} - \overline{m} = \eta_h^W & , \; h = 1, \dots, H-1 \\ \alpha_H^W = -\sum_{h=1}^{H-1} \beta_h^W = \overline{m}_{\cdot H} - \overline{m} = \eta_H^W & \end{array}\right\} \text{factor main effects}$$

$$\alpha_{g,h}^{ZW} = \dots = m_{g,h} - \overline{m}_{g\cdot} - \overline{m}_{\cdot h} + \overline{m} \qquad g = 1, \dots, G \quad h = 1, \dots, H$$

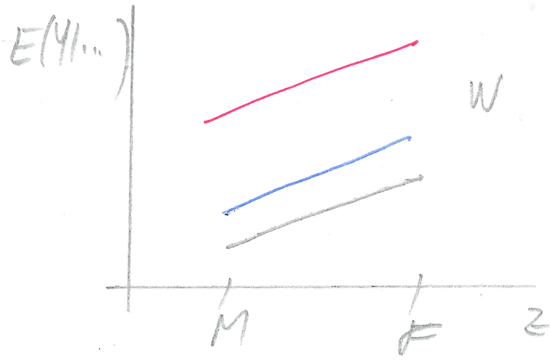
not that important

Additive model $M_{Z+W}: \sim Z + W$

- additivity in the effect of the Z and W covariates on the response expect.

$\equiv$ $\alpha_{11}^{ZW} = \dots = \alpha_{G,H}^{ZW}$ in the interaction model

$\beta^{ZW} = 0_{(G-1)\cdot(H-1)}$



Full-rank param:

$$m_{g,h} = \underbrace{\beta_0}_{1} + \underbrace{c_g^T \beta^Z}_{G-1} + \underbrace{d_h^T \beta^W}_{H-1}$$

$$\alpha_0 + \alpha_g^Z + \alpha_h^W$$

rank $= G + H - 1$

- to have this, it is sufficient to assume $n_{g\cdot} > 0 \;\forall g$
$n_{\cdot h} > 0 \;\forall h$
(some of $n_{g,h}$ might be 0)

additivity implies (partial effects)

- $\forall g_1 \neq g_2$ $\quad m_{g_1,h} - m_{g_2,h} = \overline{m}_{g_1\cdot} - \overline{m}_{g_2\cdot}$
$h = 1,\dots,H$ $\quad = \eta_{g_1}^Z - \eta_{g_2}^Z = \theta_{g_1 g_2 \cdot}$

- $\forall h_1 \neq h_2$ $\quad m_{g,h_1} - m_{g,h_2} = \overline{m}_{\cdot h_1} - \overline{m}_{\cdot h_2}$
$g = 1,\dots,G$ $\quad = \eta_{h_1}^W - \eta_{h_2}^W = \theta_{\cdot h_1, h_2}$

plots 27-28

It was derived in Section 5.4

(as with interaction model)

sum contrasts

$$\Rightarrow \alpha_0 = \beta_0 = \bar{m}$$

$$\alpha_g^Z = \beta_g^Z = \overline{m_{g\cdot}} - \bar{m} = \eta_g^Z \quad , g = 1, .., G-1$$

$$\alpha_G^Z = \beta - \sum_{g=1}^{G-1} \beta_g^Z = \overline{m_{G\cdot}} - \bar{m} = \eta_G^Z$$

$$\alpha_h^W = \beta_h^W = \overline{m_{\cdot h}} - \bar{m} = \eta_h^W \quad , h = 1, .., H-1$$

$$\alpha_H^W = -\sum_{h=1}^{H-1} \beta_h^W = \overline{m_{\cdot H}} - \bar{m} = \eta_H^W$$

Model of effect of Z only $\quad M_Z : \sim Z$ 29

$$m_{g,h} = \underset{\alpha_0}{\beta_0} + \underset{\alpha_g^Z}{\mathbb{C}_g^T \beta^Z} \quad , \quad g = 1,.., G, \quad h = 1,..,H$$

$(\equiv \beta^W = 0_{H-1}$ in M_{Z+W} model$)$

$$\text{rank} = G \quad (\text{if } n_{g\bullet} > 0 \ \forall g)$$

it implies $\quad \forall g = 1,.., G \quad m_{g,1} = \dots = m_{g,H} = \overline{m_{g\bullet}}$

$$\overline{m_{\bullet 1}} = \dots = \overline{m_{\bullet H}}$$

Model of effect of W only $\quad M_W : \sim W$ 30

$$m_{g,h} = \underset{\alpha_0}{\beta_0} + \underset{\alpha_h^W}{\mathbb{D}_h^T \beta^W} \quad , \quad g = 1,.., G, \quad h = 1,..,H$$

$(\equiv \beta^Z = 0_{G-1}$ in M_{Z+W} model$)$

$$\text{rank} = H \quad (\text{if } n_{\bullet h} > 0 \ \forall h)$$

it implies $\quad \forall h = 1,.., H \quad m_{1,h} = \dots = m_{G,h} = \overline{m_{\bullet h}}$

$$\overline{m_{1\bullet}} = \dots = \overline{m_{G\bullet}}$$

Intercept only model $\quad M_0 : \sim 1$ 31

$$m_{g,h} = \underset{\alpha_0}{\beta_0} \quad , \quad g = 1,.., G, \quad h = 1,..,H$$

$$\text{rank} = 1 \quad (\text{if } n > 0)$$

Summary (possible two-way ANOVA models)

Model	Rank	Requirement for rank
M_{ZW} : $\sim Z + W + Z{:}W$	$G \cdot H$	$n_{gh} > 0 \ \forall g, h$
M_{Z+W} : $\sim Z + W$	$G + H - 1$	$n_{g\cdot} > 0 \ \forall g$ $n_{\cdot h} > 0 \ \forall h$
M_Z : $\sim Z$	G	$n_{g\cdot} > 0 \ \forall g$
M_W : $\sim W$	H	$n_{\cdot h} > 0 \ \forall h$
M_0 : ~ 1	1	$n > 0$

$$
\begin{array}{ccccccc}
 & & M_Z & \subset & & & \\
M_0 & \cup & & & M_{Z+W} & \subset & M_{ZW} \\
 & \subset & M_W & \cup & & &
\end{array}
$$

→ under normality (BUT...)
F-test on submodels can be used to compare different structures of the two-way classified group means

13.2.3 Least squares estimation

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(obvious) notation:

$$\overline{Y}_{g,h\bullet} := \frac{1}{n_{g,h}} \sum_{j=1}^{n_{g,h}} Y_{g,h,j} \qquad g=1,\dots,G,\ h=1,\dots,H$$

$$\overline{Y}_{g\bullet} := \frac{1}{n_{g\bullet}} \sum_{h=1}^{H} \sum_{j=1}^{n_{g,h}} Y_{g,h,j} = \frac{1}{n_{g\bullet}} \sum_{h=1}^{H} n_{g,h} \overline{Y}_{g,h\bullet} \qquad g=1,\dots,G$$

$$\overline{Y}_{\bullet h} := \frac{1}{n_{\bullet h}} \sum_{g=1}^{G} \sum_{j=1}^{n_{g,h}} Y_{g,h,j} = \frac{1}{n_{\bullet h}} \sum_{g=1}^{G} n_{g,h} \overline{Y}_{g,h\bullet} \qquad h=1,\dots,H$$

$$\overline{Y} := \frac{1}{n} \sum_{g=1}^{G} \sum_{h=1}^{H} \sum_{j=1}^{n_{g,h}} Y_{g,h,j} =$$

$$= \frac{1}{n} \sum_{g=1}^{G} n_{g\bullet} \overline{Y}_{g\bullet} = \frac{1}{n} \sum_{h=1}^{H} n_{\bullet h} \overline{Y}_{\bullet h}.$$

Lemma 13.2 Least squares estimation in two-way ANOVA models

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short: Fitted values and the LSE of the group means:

(i) Interaction model M_{ZW}: $\widehat{m}_{g,h} = \widehat{Y}_{g,h,j} = \overline{Y}_{g,h\bullet}$

(ii) Additive model M_{Z+W}: $\widehat{m}_{g,h} = \widehat{Y}_{g,h,j} = \overline{Y}_{g\bullet} + \overline{Y}_{\bullet h} - \overline{Y}$

but only in case of balanced data $n_{g,h} = J \ \forall g,h$

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(iii) Model of effect of Z only M_Z: $\widehat{m}_{g,h} = \widehat{Y}_{g,h,j} = \overline{Y}_{g\bullet}$

(iv) Model of effect of W only M_W: $\widehat{m}_{g,h} = \widehat{Y}_{g,h,j} = \overline{Y}_{\bullet h}$

(v) Intercept only model $M_0: \sim 1$: $\widehat{m}_{g,h} = \widehat{Y}_{g,h,j} = \overline{Y}$.

Proof:

(i), (iii), (iv): see one-way classification (Lemma 13.1)

(v): see Lemma 7.1

- only (ii) (additive model) must be shown now $\boxed{34}$

Let us consider sum contrast parameterization for both Z and W:

$$\mathbb{C} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ -1 & \cdots & 1 \end{pmatrix} \qquad \mathbb{D} = \begin{pmatrix} 1 & \cdots & 0 \\ & & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix}$$

$$M_{Z+W}: \quad m_{g,h} = \beta_0 + c_g^T \beta^Z + d_h^T \beta^W$$

$$= \alpha_0 + \alpha_g^Z + \alpha_h^W$$

$\equiv$ 'α' parameterization where

$$\sum_g \alpha_g^Z = 0 \quad \& \quad \sum_h \alpha_h^W = 0 \quad \text{(Section 4.4)}$$

$\rightarrow$ fitted values $\hat{Y}_{g,h,j} = \hat{m}_{g,h} = \hat{\alpha}_0 + \hat{\alpha}_g^Z + \hat{\alpha}_h^W$,

where $\hat{\alpha} = (\hat{\alpha}_0, \hat{\alpha}_1^Z, \dots, \hat{\alpha}_G^Z, \hat{\alpha}_1^W, \dots, \hat{\alpha}_H^W)^T$ minimizes the sum of squares subject to two constraints $\sum_g \alpha_g^Z = 0$ & $\sum_h \alpha_h^W = 0$

$$SS(\alpha) = \sum_{g=1}^{G} \sum_{h=1}^{H} \sum_{j=1}^{J} (Y_{g,h,j} - \alpha_0 - \alpha_g^Z - \alpha_h^W)^2$$

$$\rightarrow \frac{\partial SS}{\partial \alpha} \overset{!}{=} 0$$

solve subject to constraints

(all derivatives will be multiplied by $(-\frac{1}{2})$):

$$-\frac{1}{2}\frac{\partial SS}{\partial \alpha_0}(\alpha) = \sum_g\sum_h\sum_j Y_{g,h,j} - G\cdot H\cdot J\alpha_0 - H\cdot J\left(\sum_g \alpha_g^z\right) - G\cdot J\left(\sum_h \alpha_h^w\right)$$

$$-\frac{1}{2}\frac{\partial SS}{\partial \alpha_g^z}(\alpha) = \sum_h\sum_j Y_{g,h,j} - H\cdot J\alpha_0 - H\cdot J\alpha_g^z - J\left(\sum_h \alpha_h^w\right)$$

$g=1,.., G$

$$-\frac{1}{2}\frac{\partial SS}{\partial \alpha_h^w}(\alpha) = \sum_g\sum_j Y_{g,h,j} - G\cdot J\alpha_0 - J\cdot\left(\sum_g \alpha_g^z\right) - G\cdot J\alpha_h^w$$

$h=1,.., H$

Solve now $-\frac{1}{2}\frac{\partial SS}{\partial \alpha}(\alpha) = \mathbb{0}$ subject to $\left(\sum_g \alpha_g^z\right)=0$

$\left(\sum_h \alpha_h^w\right)=0$

$\rightarrow$ we just solve (while using the constraints)

$$\sum_g\sum_h\sum_j Y_{g,h,j} - G\cdot H\cdot J\alpha_0 = 0$$

$$\sum_h\sum_j Y_{g,h,j} - HJ\alpha_0 - H\cdot J\alpha_g^z = 0 \quad , g=1,.., G$$

$$\sum_g\sum_j Y_{g,h,j} - G\cdot J\alpha_0 - G\cdot J\alpha_h^w = 0 \quad , h=1,.., H$$

$\Rightarrow$ $\hat{\alpha}_0 = \overline{Y}$

$\hat{\alpha}_g^z = \overline{Y}_{g\cdot} - \overline{Y}$

$\hat{\alpha}_h^w = \overline{Y}_{\cdot h} - \overline{Y}$

$\Rightarrow$ $\hat{m}_{g,h} = \hat{\alpha}_0 + \hat{\alpha}_g^z + \hat{\alpha}_h^w$

$= \overline{Y}_{g\cdot} + \overline{Y}_{\cdot h} - \overline{Y}$

$g=1,.., G$

$h=1,.., H$

$\boxed{\checkmark}$

Consequence of Lemma 13.2 LSE of the means of the means 36
in the interaction and the additive
model with balanced data

Short: for both additive and interaction
model with balanced data ($n_{gh} = J \;\forall g,h$)

$$\widehat{m_{g\cdot}} = \overline{Y_{g\cdot}} \quad , g = 1, \dots, G$$

$$\widehat{m_{\cdot h}} = \overline{Y_{\cdot h}} \quad , h = 1, \dots, H$$

If additionally normality is assumed then

$$\hat{m}^{z} = \begin{pmatrix} \widehat{m_{1\cdot}} \\ \vdots \\ \widehat{m_{G\cdot}} \end{pmatrix} \Big| Z, W \sim N_G \left(\underbrace{\begin{pmatrix} \overline{m_{1\cdot}} \\ \vdots \\ \overline{m_{G\cdot}} \end{pmatrix}}_{\overline{m}^{z}} , V^{z} \right),$$

$$V^{z} = \begin{pmatrix} \frac{1}{JH} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{JH} \end{pmatrix}$$ 37

analogously for $\hat{m}^{w}$ ...

Proof: We will show only LSE of $\overline{m_{g\cdot}}$:

$$\overline{m_{g\cdot}} = \frac{1}{H}\sum_{h=1}^{H} m_{g,h}$$ = linear comb. of the group means (of vector $E(Y|Z,W)$)

$\Rightarrow$ LSE of $\overline{m_{g\cdot}}$ is the corresponding linear comb. of LSE's

$$\Rightarrow \widehat{\overline{m_{g\cdot}}} = \frac{1}{H}\sum_{h=1}^{H}\widehat{m_{gh}} =$$

$$\overset{\text{interact}}{=} \frac{1}{JH}\sum_{h=1}^{H} J\overline{Y_{gh\cdot}} = \overline{Y_{g\cdot}}$$

$$\overset{\text{additive}}{=} \frac{1}{H}\sum_{h=1}^{H}(\overline{Y_{g\cdot}} + \overline{Y_{\cdot h}} - \overline{Y}) = \overline{Y_{g\cdot}} + \underbrace{\frac{1}{HG\cdot J}\sum_{h=1}^{H} G\cdot J\,\overline{Y_{\cdot h}}}_{\overline{Y}} - \overline{Y} = \overline{Y_{g\cdot}}$$

Direct calculation or unbiasedness of LSE:

$$E(\overline{Y_{g\cdot}}|Z,W) = E\left(\frac{1}{H\cdot J}\sum_h\sum_j Y_{g,h,j}\,|\,Z,W\right) \overset{\text{triv.}}{=} \overline{m_{g\cdot}}$$

Direct calculation or properties of LSE:

$$\mathrm{var}(\overline{Y_{g\cdot}}|Z,W) = \mathrm{var}\left(\frac{1}{HJ}\sum_h\sum_j Y_{g,h,j}\,|\,Z,W\right) =$$

$$\overset{\text{uncorrel. } Y_{g,h,j}}{=} \frac{1}{(HJ)^2}\sum_h\sum_j \underbrace{\mathrm{var}(Y_{g,h,j}|Z,W)}_{\sigma^2} = \frac{\sigma^2}{H\cdot J}$$

$$\mathrm{cov}(\overline{Y_{g_1\cdot}}, \overline{Y_{g_2\cdot}}|Z,W) = 0 \quad g_1 \neq g_2$$ follows from $\mathrm{var}(Y|Z,W) = \sigma^2 I_n$.

$$\Rightarrow \mathrm{var}(\widehat{m}^2|Z,W) = \sigma^2 \cdot \begin{pmatrix} \frac{1}{HJ} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{HJ}\end{pmatrix}$$

Normality $\Leftarrow$ general properties of LSE.

13.2.4 Sums of squares and ANOVA tables with balanced data

Balanced data ($n_{gh} = J \ \forall g,h$) assumed now.

Remember the fitted values:

M_{ZW}: $\hat{m}_{g,h} = \hat{Y}_{g,h,j} = \overline{Y}_{g,h\cdot}$

M_{Z+W}: $= \overline{Y}_{g\cdot} + \overline{Y}_{\cdot h} - \overline{Y}$

M_Z: $= \overline{Y}_{g\cdot}$

M_W: $= \overline{Y}_{\cdot h}$

M_0: $= \overline{Y}$

$$M_0 \subset \begin{matrix} M_Z \\ M_W \end{matrix} \subset M_{Z+W} \subset M_{ZW}$$

→ slide: all expressions are $\|\hat{Y}_1 - \hat{Y}_0\|^2$ for certain couple submodel $\subset$ model

= numerators for F-statistics

OBSERVE: $SS(Z+W/W) = SS(Z/1)$

$SS(Z+W/Z) = SS(W/1)$

NOTATION: $SS_Z := SS(Z+W/W) = SS(Z/1)$

$SS_W := SS(Z+W/Z) = SS(W/1)$

$SS_{ZW} := SS(Z+W+ZW/Z+W)$ → always comparison of two models that differ by one factor

usual

$SS_T := \sum_g \sum_h \sum_j (Y_{g,h,j} - \overline{Y})^2$

$SS_e^{ZW} := \sum_g \sum_h \sum_j (Y_{g,h,j} - \overline{Y}_{g,h\cdot})^2$

Lemma 13.3 Breakdown of the total sum of squares in a balanced two-way classification

In case of a balanced two-way classification, the following identity holds

$$SS_T = SS_Z + SS_W + SS_{ZW} + SS_e^{ZW}.$$

≡ decomposition of total variability of outcome into

between groups variab.
(a) variability due to Z (between $\bar{Y}_{g\cdot}$'s)
(b) due to W (between $\bar{Y}_{\cdot h}$'s)
(c) due to ZW (remaining between groups)

(d) residual variability (within groups)

Proof: $\underbrace{SS_Z, SS_W, SS_{ZW}}$ ≡ numerator differences in SS's for testing a series of submodels $M_0 \subset M_Z \subset M_{Z+W} \subset M_{ZW}$

SS_e^{ZW} ≡ denominator SS

- each SS = squared norm of a projection of Y into appropriate subspace of $\mathbb{R}^n$

vec-dimension: 1   G-1   H-1   . . . . .

basis of $\mathbb{R}^n$: $(\mathbb{1} \mid Q^z \mid Q^w \mid Q^{zw} \mid N)$

$\mathcal{M}_0$

$\mathcal{M}_z$

$\mathcal{M}_{z+w}$

$\mathcal{M}_{zw}$

U^0 (residuals of M_0)
= projection of Y into $\mathcal{M}(Q^z, Q^w, Q^{zw}, N)$

$$\|U^0\|^2 = SS_e^0 = SS_T$$

$$(\mathbb{1}, Q^z, Q^w, Q^{zw}, N)$$

U^0 $\|U^0\|^2 = SS_e^0 = SS_T$

D_z $\|D_z\|^2 = SS_z$

D_w $\|D_w\|^2 = SS_w$

D_{zw} $\|D_{zw}\|^2 = SS_{zw}$

U $\|U\|^2 = SS_e^{zw}$

$$U^0 = \underbrace{D_z + D_w + D_{zw} + U}_{\text{mutually orthogonal}}$$

$$\Rightarrow \|U^0\|^2 = \|D_z\|^2 + \|D_w\|^2 + \|D_{zw}\|^2 + \|U\|^2$$

$$SS_T = SS_z + SS_w + SS_{zw} + SS_e^{zw}$$

ANOVA tables

BALANCED data

→ ANOVA table type I
= ANOVA table type II

$SS_Z = SS(Z+W \mid W)$ $\qquad$ $SS_W = SS(Z+W \mid Z)$

$= SS(Z \mid 1)$ $\qquad$ $= SS(W \mid 1)$

With random covariates,
balanced data $\equiv P(Z=g, W=h) = \frac{1}{G \cdot H}$

M_Z vs. M_0 $\equiv$ marginal importance of Z
(Does $\mathbb{E}(Y \mid Z)$ depend on Z?)

M_{Z+W} vs. M_W $\equiv$ partial importance of Z
(Does $\mathbb{E}(Y \mid Z, W)$ depend on Z
for a given value of W?)

→ F-test will provide the same answer.