On the Validity of Discrete Korn's Inequality for Nonconforming Finite Element Spaces

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Abstract. We investigate the validity of discrete Korn's inequality for general nonconforming finite element spaces defined over simplicial triangulations in two and three dimensions.

1 Introduction

This paper is devoted to investigations of the validity of the discrete Korn inequality for nonconforming finite elements. The main feature of nonconforming finite elements is that the usual continuity requirement across faces of the triangulation is weakened to the validity of some patch test, which allows jumps of finite element functions across faces. These jumps cause additional difficulties in theoretical investigations of the corresponding finite element discretizations, however, there are also important advantages of this kind of finite elements. For instance, nonconforming finite elements are more suitable for a parallel implementation than conforming elements since they lead to a cheap local communication between processors. Moreover, nonconforming elements are very attractive for solving problems describing incompressible or nearly incompressible materials since they usually fulfil an inf–sup condition.

We denote by $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a bounded domain, the boundary of which consists of the parts Γ^D and Γ^N, and we define the space

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^d ;\ \mathbf{v} = \mathbf{0} \ \text{ on } \Gamma^D\} .$$

For problems from linear elasticity or fluid mechanics with surface forces prescribed on Γ^N, the Korn inequality

$$|\mathbf{v}|_{1,\Omega} \le C \, \|\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\|_{0,\Omega} \qquad \forall \ \mathbf{v} \in \mathbf{V} \tag{1}$$

is needed to prove the unique solvability of the respective problem and the continuous dependence of the solution on the data. Clearly, if the space $\mathbf{V}$ is approximated by a nonconforming finite element space $\mathbf{V}_h$, the validity of a discrete analogue of (1) (the so–called discrete Korn inequality) is also needed. Moreover, it is important that the discrete Korn inequality holds with a constant independent of the mesh parameter h since otherwise we cannot establish optimal error estimates for the discrete solutions.

In the two–dimensional case, the validity of the discrete Korn inequality was already investigated in several papers (cf. e.g. [2], [3], [6]). Recently, using an idea of [2], it was shown in [5] that, in two dimensions, the discrete Korn inequality holds for any nonconforming finite element space satisfying the patch test of order 2. The aim of the present paper is to generalize the technique used in [5] to three–dimensions. For simplicity, we only consider simplicial triangulations although in [5] triangulations consisting of both triangles and quadrilaterals were considered.

First, in the next section, we summarize notation and basic assumptions made in this paper. Then, in Section 3, we present the proof in the two–dimensional case and, finally, in Section 4, we generalize this proof to three dimensions.

2 Notation and assumptions

For simplicity, we assume that the above–introduced bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, has a polygonal resp. polyhedral boundary $\partial\Omega$. We assume that $\partial\Omega = \Gamma^D \cup \Gamma^N$, where $\Gamma^D \cap \Gamma^N = \emptyset$ and $\mathrm{meas}_{d-1}(\Gamma^D) \neq 0$.

We denote by $\mathcal{T}_h$ any triangulation of the domain Ω consisting of shape regular simplicial elements K having the usual compatibility properties (see e.g. [1]) and satisfying $h_K \equiv \mathrm{diam}(K) \leq h$ for any $K \in \mathcal{T}_h$. We assume that any face (=edge if $d = 2$) of $\mathcal{T}_h$ lying on $\partial\Omega$ belongs either to Γ^D or to Γ^N.

We denote by $\mathcal{F}_h$ the set of the faces F of $\mathcal{T}_h$ and by $\mathcal{F}_h^i$ and $\mathcal{F}_h^D$ the subsets of $\mathcal{F}_h$ consisting of inner faces and boundary faces lying on Γ^D, respectively. For any face F, we choose a fixed unit normal vector $\mathbf{n}_F$. If $F \subset \partial\Omega$, then $\mathbf{n}_F$ coincides with the outer normal vector $\mathbf{n}$ to the boundary of Ω. For any inner face $F \in \mathcal{F}_h^i$, we define the jump $[|v|]_F$ of a function v across F by

$$[|v|]_F = (v|_K)|_F - (v|_{\widetilde{K}})|_F \,,$$

where K, $\widetilde{K}$ are the two elements adjacent to F denoted in such a way that $\mathbf{n}_F$ points into $\widetilde{K}$. If an face $F \in \mathcal{F}_h$ lies on the boundary of Ω, then we set $[|v|]_F = v|_F$.

Throughout the paper we use standard notation for function spaces and norms, see e.g. [1]. We only mention that we denote by $L_0^2(G)$ the space of functions from $L^2(G)$ having zero mean value on a set G. The inner product in $L^2(G)$ will be denoted by $(\cdot,\cdot)_G$ and we set $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$.

3 Discrete Korn inequality in 2D

In this section we consider the two–dimensional case $d = 2$ and we prove that the discrete Korn inequality holds for any nonconforming finite element space which is defined over the triangulation $\mathcal{T}_h$, approximates the space $\mathbf{V}$ and satisfies the patch test of order 2. Any such space is a subspace of the space

$$\mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^2\,;\; \mathbf{v}|_K \in H^1(K)^2 \quad \forall\, K \in \mathcal{T}_h\,, \\ \int_F [|\mathbf{v}|]_F\, q \,\mathrm{d}\sigma = \mathbf{0} \quad \forall\, q \in P_1(F),\; F \in \mathcal{F}_h^i \cup \mathcal{F}_h^D\}$$

and hence it suffices to investigate the Korn inequality for this large space. This approach implies that, for all nonconforming spaces having the above–mentioned properties, the constants from the respective discrete Korn inequalities can be bounded by one fixed constant. The validity of the discrete Korn inequality for $\mathbf{V}_h$ was already proved in [5] and here we present the main ideas of the proof which will be generalized to the three–dimensional case in the next section.

The starting point of the proof of the discrete Korn inequality for the space $\mathbf{V}_h$ is the identity

$$|\nabla\,\mathbf{v}|^2 = \frac{1}{2}\,(\nabla\,\mathbf{v} + \nabla\,\mathbf{v}^\mathsf{T})\cdot(\nabla\,\mathbf{v} - \operatorname{curl}\mathbf{z}) \\ + \nabla\,\mathbf{v}\cdot\operatorname{curl}\mathbf{z} + \frac{1}{2}\,(\operatorname{rot}\mathbf{v} - \operatorname{div}\mathbf{z})\operatorname{rot}\mathbf{v}\,, \qquad (2)$$

where

$$\operatorname{rot}\mathbf{v} = -\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\,, \qquad \operatorname{curl}\mathbf{z} = \begin{pmatrix} \dfrac{\partial z_1}{\partial x_2} & -\dfrac{\partial z_1}{\partial x_1} \\ \dfrac{\partial z_2}{\partial x_2} & -\dfrac{\partial z_2}{\partial x_1} \end{pmatrix}.$$

It is easy to verify that the identity (2) holds for any vector functions $\mathbf{v}$ and $\mathbf{z}$. We choose $\mathbf{v} \in \mathbf{V}_h$ and we try to find a function $\mathbf{z}$ such that, after integrating (2) over elements and summing up, the second and third terms on the right–hand side of (2) vanish. Let us start with the second term, i.e., let us construct a space $\mathbf{Z}_h$ (as large as possible) such that the orthogonality relation

$$\sum_{K\in\mathcal{T}_h} (\nabla\,\mathbf{v}, \operatorname{curl}\mathbf{z})_K = 0 \qquad \forall\;\mathbf{v}\in\mathbf{V}_h,\;\mathbf{z}\in\mathbf{Z}_h \tag{3}$$

holds.

For any element K, let $\mathbf{n}_{\partial K} = (n_{\partial K1}, n_{\partial K2})$ be the unit outer normal vector to the boundary of K. We denote by $\mathbf{t}_{\partial K} = (-n_{\partial K2}, n_{\partial K1})$ the corresponding tangent vector. Since

$$\begin{aligned}\nabla\,\mathbf{v}\cdot\operatorname{curl}\mathbf{z} = &\left[\frac{\partial}{\partial x_1}\left(v_1\,\frac{\partial z_1}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(v_1\,\frac{\partial z_1}{\partial x_1}\right)\right]\\ &+\left[\frac{\partial}{\partial x_1}\left(v_2\,\frac{\partial z_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(v_2\,\frac{\partial z_2}{\partial x_1}\right)\right],\end{aligned} \tag{4}$$

it follows from the Gauss integral theorem that

$$(\nabla\,\mathbf{v}, \operatorname{curl}\mathbf{z})_K = \int_{\partial K} v_1\,\frac{\partial z_1}{\partial \mathbf{t}_{\partial K}} + v_2\,\frac{\partial z_2}{\partial \mathbf{t}_{\partial K}}\,\mathrm{d}\sigma \quad \forall\;\mathbf{v}\in H^1(K)^2,\;\mathbf{z}\in H^2(K)^2\,. \tag{5}$$

For $\mathbf{z}\in C_0^\infty(K)^2$, the right–hand side of (5) vanishes and since the space $C_0^\infty(K)$ is dense in $H_0^1(K)$, we infer that

$$(\nabla\,\mathbf{v}, \operatorname{curl}\mathbf{z})_K = 0 \qquad \forall\;\mathbf{v}\in H^1(K)^2,\;\mathbf{z}\in H_0^1(K)^2\,.$$

Thus, the relation in (3) holds for any $\mathbf{z}$ from the space

$$\mathbf{G}_h = \{\mathbf{v}\in H_0^1(\Omega)^2\,;\;\mathbf{v}|_K\in H_0^1(K)^2\;\;\forall\;K\in\mathcal{T}_h\}\,.$$

If $\mathbf{z}\in C(\overline{\Omega})$ is a piecewise smooth function, then the derivatives of $\mathbf{z}$ along edges are continuous across edges and hence we deduce using (5) that

$$\sum_{K\in\mathcal{T}_h} (\nabla\,\mathbf{v}, \operatorname{curl}\mathbf{z})_K = \sum_{F\in\mathcal{F}_h}\int_F [|v_1|]_F\,\frac{\partial z_1}{\partial \mathbf{t}_F} + [|v_2|]_F\,\frac{\partial z_2}{\partial \mathbf{t}_F}\,\mathrm{d}\sigma\,,$$

where $\mathbf{t}_F = (-n_{F2}, n_{F1})$ with n_{F1}, n_{F2} being the components of $\mathbf{n}_F$. Taking into account the properties of the space $\mathbf{V}_h$, we see that the relation in (3) also holds for any $\mathbf{z}$ from the space

$$\mathbf{R}_h = \{\mathbf{v}\in C(\overline{\Omega})^2\,;\;\mathbf{v}|_K\in P_2(K)^2\;\;\forall\;K\in\mathcal{T}_h\,,\quad \mathbf{v}=\mathbf{0}\;\text{ on }\Gamma^N\}\,.$$

Therefore, we can conclude that (3) is satisfied for

$$\mathbf{Z}_h = \mathbf{R}_h\oplus\mathbf{G}_h\,. \tag{6}$$

Now the question is whether one can find $\mathbf{z} \in \mathbf{Z}_h$ such that

$$\sum_{K \in \mathcal{T}_h} (\operatorname{rot} \mathbf{v} - \operatorname{div} \mathbf{z}, \operatorname{rot} \mathbf{v})_K = 0 \,. \tag{7}$$

This is indeed possible since one can prove (cf. [5]) the validity of the inf–sup condition

$$\sup_{\mathbf{v} \in \mathbf{Z}_h \setminus \{\mathbf{0}\}} \frac{(\operatorname{div} \mathbf{v}, q)}{|\mathbf{v}|_{1,\Omega}} \geq \beta \, \|q\|_{0,\Omega} \qquad \forall \; q \in L^2(\Omega) \tag{8}$$

with a constant $\beta > 0$ independent of h. A consequence of (8) is (cf. [4]) that, for any $\mathbf{v} \in \mathbf{V}_h$, there exists $\mathbf{z} \in \mathbf{Z}_h$ such that

$$(\operatorname{div} \mathbf{z}, q) = \sum_{K \in \mathcal{T}_h} (\operatorname{rot} \mathbf{v}, q)_K \qquad \forall \; q \in L^2(\Omega) \,, \tag{9}$$

$$|\mathbf{z}|_{1,\Omega} \leq \frac{\sqrt{2}}{\beta} \sqrt{\sum_{K \in \mathcal{T}_h} |\mathbf{v}|^2_{1,K}} \,. \tag{10}$$

The relation (9) implies (7) and hence it follows from (2), (3) and (7) that, for any $\mathbf{v} \in \mathbf{V}_h$, there exists $\mathbf{z} \in \mathbf{Z}_h$ satisfying (10) and

$$\begin{aligned} 2 \sum_{K \in \mathcal{T}_h} |\mathbf{v}|^2_{1,K} &= \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}, \nabla \mathbf{v} - \operatorname{curl} \mathbf{z})_K \\ &\leq \sqrt{\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\|^2_{0,K}} \sqrt{\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} - \operatorname{curl} \mathbf{z}\|^2_{0,K}} \,. \end{aligned}$$

In view of (10), we have

$$\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} - \operatorname{curl} \mathbf{z}\|^2_{0,K} \leq \frac{2\,\beta^2 + 4}{\beta^2} \sum_{K \in \mathcal{T}_h} |\mathbf{v}|^2_{1,K}$$

and hence we deduce that there exists a constant C independent of h such that

$$\sum_{K \in \mathcal{T}_h} |\mathbf{v}|^2_{1,K} \leq C \sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\|^2_{0,K} \qquad \forall \; \mathbf{v} \in \mathbf{V}_h \,, \tag{11}$$

which is the desired discrete Korn inequality. Note that (11) implies the validity of the Korn inequality (1) since $\mathbf{V} \subset \mathbf{V}_h$.

4 Discrete Korn inequality in 3D

In this section we generalize the techniques used in the preceding section to the three–dimensional case. Our aim is to find a 'small' space $P(F)$ such that the space

$$\mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^3\,;\ \mathbf{v}|_K \in H^1(K)^3 \ \ \forall\, K \in \mathcal{T}_h\,,$$
$$\int_F [|\mathbf{v}|]_F\, q\,\mathrm{d}\sigma = \mathbf{0} \ \ \forall\, q \in P(F),\ F \in \mathcal{F}_h^i \cup \mathcal{F}_h^D\}$$

satisfies the discrete Korn inequality (11) with C independent of h.

First let us look for a three–dimensional analogue of the relation (2). We replace the two–dimensional tensor $\mathrm{curl}\,\mathbf{z}$ by a suitable three–dimensional tensor $\sigma(\mathbf{z})$ depending on a function $\mathbf{z}$. For any tensor $\sigma(\mathbf{z})$, we have

$$\begin{aligned} |\nabla\,\mathbf{v}|^2 \ &= \ \frac{1}{2}\,(\nabla\,\mathbf{v} + \nabla\,\mathbf{v}^\mathsf{T})\cdot(\nabla\,\mathbf{v} - \sigma(\mathbf{z})) + \nabla\,\mathbf{v}\cdot\sigma(\mathbf{z}) \\ &\qquad + \frac{1}{2}\,(\nabla\,\mathbf{v} - \nabla\,\mathbf{v}^\mathsf{T})\cdot(\nabla\,\mathbf{v} - \sigma(\mathbf{z}))\,. \end{aligned}$$

Since

$$(\nabla\,\mathbf{v} - \nabla\,\mathbf{v}^\mathsf{T})\cdot\nabla\,\mathbf{v} = \frac{1}{2}\,(\nabla\,\mathbf{v} - \nabla\,\mathbf{v}^\mathsf{T})\cdot(\nabla\,\mathbf{v} - \nabla\,\mathbf{v}^\mathsf{T}) = |\mathrm{rot}\,\mathbf{v}|^2\,, \qquad (12)$$

where

$$\mathrm{rot}\,\mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3},\quad \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1},\quad \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\,,$$

we can require that

$$(\nabla\,\mathbf{v} - \nabla\,\mathbf{v}^\mathsf{T})\cdot(\nabla\,\mathbf{v} - \sigma(\mathbf{z})) = (\mathrm{rot}\,\mathbf{v} - \mathrm{div}\,\mathbf{z})\cdot\mathrm{rot}\,\mathbf{v}\,. \qquad (13)$$

However, $\mathrm{rot}\,\mathbf{v}$ is now a vector and hence the term $\mathrm{div}\,\mathbf{z}$ also has to be a vector. Consequently, $\mathbf{z}$ has to be a tensor, i.e., $\mathbf{z} = \{z_{ij}\}_{i,j=1}^3$. Let us recall that

$$(\mathrm{div}\,\mathbf{z})_i = \sum_{j=1}^3 \frac{\partial z_{ij}}{\partial x_j}\,, \qquad i = 1, 2, 3\,.$$

From (12) and (13), it follows that

$$(\nabla\,\mathbf{v} - \nabla\,\mathbf{v}^\mathsf{T})\cdot\sigma(\mathbf{z}) = \mathrm{div}\,\mathbf{z}\cdot\mathrm{rot}\,\mathbf{v}\,.$$

This is fulfilled if and only if

$$\sigma(\mathbf{z}) = \frac{1}{2}\,\lambda(\mathbf{z}) + \gamma(\mathbf{z}) + \delta(\mathbf{z})\,,$$

where

$$\lambda(\mathbf{z}) = \begin{pmatrix} 0 & -\dfrac{\partial z_{33}}{\partial x_3} & \dfrac{\partial z_{22}}{\partial x_2} \\ \dfrac{\partial z_{33}}{\partial x_3} & 0 & -\dfrac{\partial z_{11}}{\partial x_1} \\ -\dfrac{\partial z_{22}}{\partial x_2} & \dfrac{\partial z_{11}}{\partial x_1} & 0 \end{pmatrix},$$

$$\gamma(\mathbf{z}) = \begin{pmatrix} 0 & -\dfrac{\partial z_{31}}{\partial x_1} & \dfrac{\partial z_{21}}{\partial x_1} \\ \dfrac{\partial z_{32}}{\partial x_2} & 0 & -\dfrac{\partial z_{12}}{\partial x_2} \\ -\dfrac{\partial z_{23}}{\partial x_3} & \dfrac{\partial z_{13}}{\partial x_3} & 0 \end{pmatrix}$$

and $\delta(\mathbf{z})$ is any symmetric tensor depending on $\mathbf{z}$. This tensor will be chosen in such a way that there exists a 'large' space $\mathbf{Z}_h$ satisfying

$$\sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v}, \sigma(\mathbf{z}))_K = 0 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h,\ \mathbf{z} \in \mathbf{Z}_h\,, \tag{14}$$

which is a counterpart of (3). To achieve this we look for such $\delta(\mathbf{z})$ that the term $(\nabla \mathbf{v}, \sigma(\mathbf{z}))_K$ can be written in an analogous form as in (5) (now, of course, with two independent tangential vectors). Therefore, like in (4), we want that the product $\nabla \mathbf{v} \cdot \sigma(\mathbf{z})$ can be written as the sum of terms of the type

$$\frac{\partial}{\partial x_i}\left(v_l\,\frac{\partial z_{mn}}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(v_l\,\frac{\partial z_{mn}}{\partial x_i}\right), \qquad i \neq j\,. \tag{15}$$

Integrating this expression over an element K and applying the Gauss integral theorem, we get $\int_{\partial K} v_l\, \mathbf{t} \cdot \nabla z_{mn}\, \mathrm{d}\sigma$, where $\mathbf{t}$ is a tangential vector to ∂K. The tensor $\delta(\mathbf{z})$ can be defined as $\frac{1}{2}\,\mu(\mathbf{z}) + \kappa(\mathbf{z})$, where $\mu(\mathbf{z})$, $\kappa(\mathbf{z})$ are symmetric tensors such that both the products $\nabla \mathbf{v} \cdot (\lambda(\mathbf{z}) + \mu(\mathbf{z}))$ and $\nabla \mathbf{v} \cdot (\gamma(\mathbf{z}) + \kappa(\mathbf{z}))$ can be written as sums of terms of the type (15). We obtain

$$\mu(\mathbf{z}) = \begin{pmatrix} 0 & \dfrac{\partial z_{11}}{\partial x_3} - \dfrac{\partial z_{22}}{\partial x_3} & \dfrac{\partial z_{33}}{\partial x_2} - \dfrac{\partial z_{11}}{\partial x_2} \\ \dfrac{\partial z_{11}}{\partial x_3} - \dfrac{\partial z_{22}}{\partial x_3} & 0 & \dfrac{\partial z_{22}}{\partial x_1} - \dfrac{\partial z_{33}}{\partial x_1} \\ \dfrac{\partial z_{33}}{\partial x_2} - \dfrac{\partial z_{11}}{\partial x_2} & \dfrac{\partial z_{22}}{\partial x_1} - \dfrac{\partial z_{33}}{\partial x_1} & 0 \end{pmatrix},$$

$$\kappa(\mathbf{z}) = \mathrm{diag}\left(\frac{\partial z_{31}}{\partial x_2} - \frac{\partial z_{21}}{\partial x_3},\ \frac{\partial z_{12}}{\partial x_3} - \frac{\partial z_{32}}{\partial x_1},\ \frac{\partial z_{23}}{\partial x_1} - \frac{\partial z_{13}}{\partial x_2}\right).$$

Thus, setting

$$\sigma(\mathbf{z}) = \frac{1}{2}\,(\lambda(\mathbf{z}) + \mu(\mathbf{z})) + \gamma(\mathbf{z}) + \kappa(\mathbf{z})\,,$$

we deduce in the same way as in the preceding section, that (14) holds for $\mathbf{Z}_h$ defined by (6) with

$$\begin{aligned}\mathbf{G}_h &= \{\mathbf{v} \in H^1_0(\Omega)^{3\times 3}\,;\ \mathbf{v}|_K \in H^1_0(K)^{3\times 3}\ \ \forall\, K \in \mathcal{T}_h\}\,,\\ \mathbf{R}_h &= \{\mathbf{v} \in C(\overline{\Omega})^{3\times 3}\,;\ \mathbf{v}|_K \in R(K)^{3\times 3}\ \ \forall\, K \in \mathcal{T}_h\,,\quad \mathbf{v} = \mathbf{0}\ \text{ on } \Gamma^N\}\,,\end{aligned}$$

where $R(K) \subset H^1(K)$ is a space consisting of smooth functions which do not belong to $H^1_0(K)$ (except for 0) and whose tangential derivatives on the faces $F \subset \partial K$ belong to the spaces $P(F)$ from the definition of $\mathbf{V}_h$.

Further, we again have to investigate whether we can find $\mathbf{z} \in \mathbf{Z}_h$ such that

$$\sum_{K\in\mathcal{T}_h} (\operatorname{rot}\mathbf{v} - \operatorname{div}\mathbf{z}, \operatorname{rot}\mathbf{v})_K = \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{3} \Big((\operatorname{rot}\mathbf{v})_i - (\operatorname{div}\mathbf{z})_i, (\operatorname{rot}\mathbf{v})_i\Big)_K = 0\,.$$

Since $(\operatorname{div}\mathbf{z})_i$ is the divergence of the i–th row of $\mathbf{z}$, we see that it is sufficient when the spaces

$$\{\mathbf{v} \in H^1(\Omega)^3\,;\ \mathbf{v}|_K \in [H^1_0(K) \oplus R(K)]^3\ \ \forall\, K \in \mathcal{T}_h\,,\quad \mathbf{v} = \mathbf{0}\ \text{ on } \Gamma^N\}$$

and $L^2(\Omega)$ satisfy the inf–sup condition (8) with a constant $\beta > 0$ independent of h. The validity of this inf–sup condition can be shown using the techniques of [5] and [4] if we set

$$R(K) = P_1(K) \oplus \operatorname{span}\{\lambda_1\,\lambda_2\,\lambda_3,\ \lambda_1\,\lambda_2\,\lambda_4,\ \lambda_1\,\lambda_3\,\lambda_4,\ \lambda_2\,\lambda_3\,\lambda_4\}\,,$$

where λ_1, λ_2, λ_3, λ_4 are the barycentric coordinates on K, and if we assume that, on any element K, the inf–sup condition for the spaces $H^1_0(K)^3$ and $L^2_0(K)$ holds with a constant independent of K (this is the case e.g. if all elements are similar with a finite number of reference tetrahedra).

Now, the spaces $P(F)$ containing the tangential derivatives of functions from $R(K)$ on faces $F \subset \partial K$ can be defined by

$$P(F) = \operatorname{span}\{1,\ \lambda_1\,(\lambda_2 - \lambda_3), \lambda_2\,(\lambda_3 - \lambda_1)\}\,,$$

where λ_1, λ_2, λ_3 are the barycentric coordinates on F. Thus, using this space in the definition of the space $\mathbf{V}_h$ and applying the techniques from the preceding section, we deduce that the discrete Korn inequality (11) holds with C independent of h. Particularly, we see that, in the three–dimensional case, the discrete Korn inequality holds for any nonconforming finite element space approximating the space $\mathbf{V}$ and satisfying the patch test of order 3.

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