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On Korn's Inequality for Nonconforming Finite Elements

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We investigate the validity of discrete Korn's inequality for general nonconforming finite element spaces defined over triangulations consisting of triangles and/or quadrilaterals. We show that for spaces satisfying the patch test of order 2 the discrete Korn inequality holds whereas it fails for spaces satisfying the patch test of order 1 only.

1 Introduction

Since the pioneering work by Crouzeix et al. (1973) many various nonconforming finite elements have been developed, cf. e.g. Fortin et al. (1983), Fortin (1985), Crouzeix et al. (1989), Rannacher et al. (1992), Kouhia et al. (1993), Cai et al. (1999), Knobloch et al. (1999). The main feature of nonconforming finite elements is that the usual continuity requirement across edges of the triangulation is weakened to the validity of some patch test, which allows jumps of finite element functions across edges. Although these jumps cause additional difficulties in theoretical investigations of the corresponding finite element discretizations, the application of nonconforming elements can be justified since they possess several favourable properties. First, nonconforming finite elements are more suitable for a parallel implementation than conforming elements since they lead to a cheap local communication when the method is implemented on a MIMD machine. Nowadays, with the increasing importance of parallel computers for scientific computations, this feature becomes still more and more important. Another important feature of nonconforming finite elements is that they usually fulfil an inf–sup condition so that they are very attractive for solving problems describing incompressible or nearly incompressible materials.

As a model problem let us consider the following boundary value problem of plain linear elasticity (cf. Nečas et al., 1981):

$$
\begin{aligned}
-\mu\,\Delta\,\mathbf{u}-(\lambda+\mu)\,\nabla\operatorname{div}\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega && (1)\\
\mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma^D && (2)\\
\lambda\,(\operatorname{div}\mathbf{u})\,\mathbf{n}+\mu\,(\nabla\,\mathbf{u}+\nabla\,\mathbf{u}^{\mathsf{T}})\,\mathbf{n} &= \mathbf{g} \quad \text{on } \Gamma^N && (3)
\end{aligned}
$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain whose boundary is Lipschitz–continuous and consists of two disjoint parts Γ^D and Γ^N such that $\operatorname{meas}_1(\Gamma^D)\neq 0$. The domain Ω represents the configuration of an elastic isotropic body in the absence of forces and the vector $\mathbf{u}$ describes the displacements of the points of Ω under the influence of the volume force $\mathbf{f}$ and the surface force $\mathbf{g}$. The material parameters λ, $\mu>0$ are the so–called Lamé coefficients and the vector $\mathbf{n}$ is the outer normal vector to the boundary of Ω. We define the space

$$\mathbf{V}=\{\mathbf{v}\in H^1(\Omega)^2\;:\;\mathbf{v}=\mathbf{0}\;\text{ on }\Gamma^D\} \tag{4}$$

where $H^1(\Omega)$ is the Sobolev space of square integrable functions in Ω whose generalized first derivatives are also square integrable in Ω. Then the weak formulation of the above problem reads: Find $\mathbf{u}\in\mathbf{V}$ such that

$$\int_\Omega \lambda\,(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v})+\frac{\mu}{2}\,(\nabla\,\mathbf{u}+\nabla\,\mathbf{u}^{\mathsf{T}})\cdot(\nabla\,\mathbf{v}+\nabla\,\mathbf{v}^{\mathsf{T}})\,\mathrm{d}x=\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x+\int_{\Gamma^N}\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\sigma \qquad \forall\,\mathbf{v}\in\mathbf{V} \tag{5}$$

The left–hand side represents a continuous bilinear form on $\mathbf{V}\times\mathbf{V}$ which is $\mathbf{V}$–elliptic in consequence of the so–called Korn inequality (cf. Nečas et al., 1981)

$$|\mathbf{v}|_{1,\Omega}\leq C\,\|\nabla\,\mathbf{v}+\nabla\,\mathbf{v}^{\mathsf{T}}\|_{0,\Omega} \qquad \forall\,\mathbf{v}\in\mathbf{V} \tag{6}$$

Therefore, there exists a unique weak solution. If we approximate this weak solution using the finite element method, we need the ellipticity (uniform with respect to the discretization parameter) of the respective bilinear form on the finite element space $\mathbf{V}_h$ approximating $\mathbf{V}$ since otherwise we cannot guarantee the unique solvability and optimal convergence properties for the discrete problem. In the case of conforming finite element methods the uniform ellipticity of the bilinear form is a direct consequence of (6) since $\mathbf{V}_h \subset \mathbf{V}$. However, nonconforming finite element spaces approximating $\mathbf{V}$ are not contained in $\mathbf{V}$ and hence it is not obvious whether a discrete analogue of (6) holds and whether it holds with a constant C independent of the discretization parameter. Let us remark that the validity of (6) is also necessary for solving other types of problems than above, e.g., for solving fluid mechanical problems with surface forces prescribed on a part of the boundary (cf. Knobloch et al., 1998).

The validity of a discrete analogue of (6) has been already investigated for several nonconforming finite element spaces (cf. e.g. Falk et al., 1990; Falk, 1991; Kouhia et al., 1993) and it is known that some of the nonconforming spaces satisfy the discrete Korn inequality whereas for some other spaces the inequality only holds with a constant C depending on the discretization parameter or it does not hold at all. However, general conditions on nonconforming finite element spaces assuring the validity of the discrete Korn inequality are not available in the literature.

Therefore, in this paper, we investigate the validity of a discrete version of (6) for general nonconforming finite element spaces $\mathbf{V}_h$ approximating the space $\mathbf{V}$. We consider triangulations of Ω consisting of triangles and/or quadrilaterals and we show that the discrete Korn inequality holds with a constant C independent of h whenever the space $\mathbf{V}_h$ satisfies the patch test of order 2. Further, we show that the validity of the discrete Korn inequality cannot be expected of spaces which satisfy the patch test of order 1 only. The proof of the discrete Korn inequality relies on a technique developed in Falk (1991) and uses a suitable inf–sup condition.

The paper is organized in the following way. In the next section we summarize notation and all the assumptions made in this paper. In Section 3 we prove an inf–sup condition which is the most important tool for proving the discrete Korn inequality in Section 4. Finally, in Section 5, we give examples of general nonconforming first order spaces for which the discrete Korn inequality does not hold.

2 Notation and Assumptions

For simplicity we assume that the above–introduced bounded domain $\Omega \subset \mathbb{R}^2$ has a polygonal boundary $\partial\Omega$. We assume that $\partial\Omega = \Gamma^D \cup \Gamma^N$, where $\Gamma^D \cap \Gamma^N = \emptyset$ and $\mathrm{meas}_1(\Gamma^D) \neq 0$.

We denote by $\mathcal{T}_h$ any triangulation of the domain Ω consisting of open triangular and/or quadrilateral elements K having the usual compatibility properties (see e.g. Ciarlet, 1991) and satisfying $h_K \equiv \mathrm{diam}(K) \leq h$ for any $K \in \mathcal{T}_h$. We assume that any edge of $\mathcal{T}_h$ lying on $\partial\Omega$ belongs either to Γ^D or to Γ^N.

We denote by $\mathcal{T}_h^*$ a triangulation obtained from $\mathcal{T}_h$ by dividing each quadrilateral element of $\mathcal{T}_h$ into two triangles. This construction of $\mathcal{T}_h^*$ is not unique unless $\mathcal{T}_h$ only consists of triangles in which case we have $\mathcal{T}_h^* = \mathcal{T}_h$. Thus, we assume that, for each triangulation $\mathcal{T}_h$, one of the possible triangulations $\mathcal{T}_h^*$ has been fixed. We require that all the triangulations under consideration are such that there exists a constant σ independent of h satisfying

$$\frac{h_K}{\varrho_K} \leq \sigma \qquad \forall\, K \in \mathcal{T}_h^* \tag{7}$$

where ϱ_K is the maximum diameter of circles inscribed into K. Note that our assumptions do not exclude the case when some quadrilateral elements of $\mathcal{T}_h$ degenerate to triangles.

We denote by $\mathcal{E}_h$ the set of the edges E of $\mathcal{T}_h$ and by $\mathcal{E}_h^i$, $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$ the subsets of $\mathcal{E}_h$ consisting of inner edges, boundary edges lying on Γ^D and boundary edges lying on Γ^N, respectively. Further, we denote by $\mathcal{E}_h^*$ the set of the edges of $\mathcal{T}_h^*$. For any edge E, we choose a fixed unit normal vector $\mathbf{n}_E = (n_{E1}, n_{E2})$ and we introduce a tangent vector $\mathbf{t}_E = (-n_{E2}, n_{E1})$. If $E \subset \partial\Omega$, then $\mathbf{n}_E$ coincides with the outer normal vector $\mathbf{n}$ to the boundary of Ω.

For any inner edge $E \in \mathcal{E}_h^i$, we define the jump $[|v|]_E$ of a function v across E and the average $\langle|v|\rangle_E$ of v on E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E \qquad\qquad \langle|v|\rangle_E = \frac{1}{2}\left[(v|_K)|_E + (v|_{\widetilde{K}})|_E\right] \tag{8}$$

where K, $\widetilde{K}$ are the two elements adjacent to E denoted in such a way that $\mathbf{n}_E$ points into $\widetilde{K}$. If an edge $E \in \mathcal{E}_h$ lies on the boundary of Ω, then we set

$$[|v|]_E = \langle|v|\rangle_E = v|_E \tag{9}$$

Throughout the paper we use standard notation $L^2(G)$, $H^k(G) = W^{k,2}(G)$, $P_k(G)$, $C(\overline{G})$, etc. for the usual function spaces defined on a set $G \subset \mathbb{R}^2$, see e.g. Ciarlet (1991). We only mention that we denote by $L_0^2(G)$ the space of functions from $L^2(G)$ having zero mean value on G. The norm and seminorm in the Sobolev space $H^k(G)$ will be denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. The inner product in $L^2(G)$ will be denoted by $(\cdot,\cdot)_G$ and we set $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$.

3 An Inf–Sup Condition

To investigate the uniform validity of a discrete analogue of the Korn inequality (6), we shall need an inf–sup condition of the type

$$\sup_{\mathbf{v}\in\mathbf{Z}_h\setminus\{\mathbf{0}\}} \frac{(\operatorname{div}\mathbf{v}, q)}{|\mathbf{v}|_{1,\Omega}} \geq \beta\,\|q\|_{0,\Omega} \qquad \forall\, q \in Q \tag{10}$$

where $\mathbf{Z}_h \subset H^1(\Omega)^2$ consists of functions vanishing on Γ^N and being low degree polynomials along the edges of $\mathcal{T}_h$. The space $Q \subset L^2(\Omega)$ has to contain piecewise constant functions, which implies that the functions from $\mathbf{Z}_h$ have to be at least quadratic along the edges. Thus, it is natural to set

$$\mathbf{Z}_h = \mathbf{R}_h \oplus \mathbf{G}_h \tag{11}$$

where

$$\mathbf{R}_h = \{\mathbf{v} \in C(\overline{\Omega})^2 \,:\, \mathbf{v}|_K \in P_2(K)^2 \;\; \forall\, K \in \mathcal{T}_h^*\,, \;\; \mathbf{v} = \mathbf{0} \;\text{ on } \Gamma^N\} \tag{12}$$

$$\mathbf{G}_h = \{\mathbf{v} \in H_0^1(\Omega)^2 \,:\, \mathbf{v}|_K \in H_0^1(K)^2 \;\; \forall\, K \in \mathcal{T}_h^*\} \tag{13}$$

Then, the inf–sup condition (10) holds with $Q = L^2(\Omega)$ and $\beta > 0$ independent of h, which will be proven in this section. The triangulation $\mathcal{T}_h^*$ is used instead of $\mathcal{T}_h$ in order to simplify the proof.

First, let us formulate some general results. We denote by V and Q two real Hilbert spaces with the norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. The inner product in Q will be denoted by $(\cdot,\cdot)_Q$. Further, we introduce a continuous bilinear form $b : V \times Q \to \mathbb{R}$ satisfying

$$b(v,q) \leq \kappa\,\|v\|_V\,\|q\|_Q \qquad \forall\, v \in V,\, q \in Q \tag{14}$$

Then the following assertions hold true.

Lemma 1 *Let there exist $\beta > 0$ such that*

$$\sup_{v\in V\setminus\{0\}} \frac{b(v,q)}{\|v\|_V} \geq \beta\,\|q\|_Q \qquad \forall\, q \in Q \tag{15}$$

Then, for any continuous linear functional $g \in Q'$, there exists a function $v \in V$ such that

$$b(v,q) = \langle g, q\rangle \qquad \forall\, q \in Q \qquad\qquad \textit{and} \qquad\qquad \|v\|_V \leq \frac{1}{\beta}\,\|g\|_{Q'} \tag{16}$$

Proof. See Girault et al. (1986), p. 58, Lemma 4.1.

Corollary 1 *Let the inf–sup condition (15) hold. Then, for any $q \in Q$, there exists $v \in V$ such that*

$$b(v,q) = \|q\|_Q^2 \qquad \textit{and} \qquad \|v\|_V \le \frac{1}{\beta}\,\|q\|_Q \tag{17}$$

Proof. Setting $\langle g, \widetilde{q}\rangle = (q, \widetilde{q})_Q$ for any $\widetilde{q} \in Q$, we have $g \in Q'$ with $\|g\|_{Q'} = \|q\|_Q$ and (17) immediately follows from (16).

Lemma 2 *Let $v_1, v_2 \in V$, $q_1, q_2 \in Q$ satisfy*

$$b(v_i, q_i) = \|q_i\|_Q^2 \qquad \|v_i\|_V \le \gamma_i\,\|q_i\|_Q \qquad i = 1, 2 \tag{18}$$

$$b(v_2, q_1) = 0 \qquad (q_1, q_2)_Q = 0 \tag{19}$$

Then, denoting $q = q_1 + q_2$, there exists $v \in V$ satisfying

$$b(v,q) = \|q\|_Q^2 \qquad \|v\|_V \le C\,\|q\|_Q \tag{20}$$

where C depends only on κ, γ_1 and γ_2.

Proof. First, let us note that the second equality in (19) implies that

$$\|q\|_Q^2 = \|q_1\|_Q^2 + \|q_2\|_Q^2 \tag{21}$$

Further, according to (14) and (18), we have

$$|b(v_1, q_2)| \le \kappa\,\gamma_1\,\|q_1\|_Q\,\|q_2\|_Q \le \frac{1}{2}\,\|q_1\|_Q^2 + \frac{\kappa^2\,\gamma_1^2}{2}\,\|q_2\|_Q^2 \tag{22}$$

Thus, for $\overline{v} = v_1 + \alpha\,v_2$ with $\alpha = (1 + \kappa^2\,\gamma_1^2)/2$, we derive

$$b(\overline{v}, q) \ge \|q_1\|_Q^2 - |b(v_1, q_2)| + \alpha\,\|q_2\|_Q^2 \ge \frac{1}{2}\,\|q\|_Q^2 \tag{23}$$

Hence the first part of (20) holds for $v = \xi\,\overline{v}$ with some $\xi \in (0, 2]$. The second part of (20) immediately follows from (18) and (21).

In what follows, we shall prove two auxiliary results from which we then obtain the desired inf–sup condition by applying Lemma 2. The general results of Corollary 1 and Lemma 2 will always be used with $V \subset [H^1(G) \setminus \{1\}]^2$, $Q \subset L^2(G)$, $\|\cdot\|_V = |\cdot|_{1,G}$, $\|\cdot\|_Q = \|\cdot\|_{0,G}$ and $b(\cdot,\cdot) = (\operatorname{div}\cdot,\cdot)_G$, where $G \subset \mathbb{R}^2$ is a suitable set. Note that (14) then holds with $\kappa = \sqrt{2}$.

Lemma 3 *Let*

$$\bar{Q}_h = \{q \in L^2(\Omega) \;:\; q|_K \in P_0(K) \;\;\forall\, K \in \mathcal{T}_h^*\} \tag{24}$$

Then, for any $\overline{q} \in \bar{Q}_h$, there exists $\overline{\mathbf{v}} \in \mathbf{R}_h$ such that

$$(\operatorname{div}\overline{\mathbf{v}}, \overline{q}) = \|\overline{q}\|_{0,\Omega}^2 \qquad |\overline{\mathbf{v}}|_{1,\Omega} \le C\,\|\overline{q}\|_{0,\Omega} \tag{25}$$

where C depends only on Ω, Γ^D and σ from (7).

Proof. We denote by $\mathbf{u} \in C^\infty(\overline{\Omega})^2$ an arbitrary but fixed function such that $\int_{\Gamma^D} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\sigma = 1$ and $\mathbf{u} = \mathbf{0}$ on Γ^N. Further, we denote

$$\overline{\mathbf{R}}_h = \{\mathbf{v} \in C(\overline{\Omega})^2 \;:\; \mathbf{v}|_K \in P_2(K)^2 \;\;\forall\, K \in \mathcal{T}_h^*\} \tag{26}$$

and we define an operator $r_h \in \mathcal{L}(H^2(\Omega)^2, \overline{\mathbf{R}}_h)$ such that, for any $\mathbf{v} \in H^2(\Omega)^2$,

$$r_h\mathbf{v} = \mathbf{v} \quad \text{at any vertex of } \mathcal{T}_h^* \qquad \int_E \mathbf{v} - r_h\mathbf{v}\,\mathrm{d}\sigma = \mathbf{0} \qquad \forall\, E \in \mathcal{E}_h^* \tag{27}$$

In a standard way (cf. Ciarlet, 1991, Section 15), we obtain

$$|r_h \mathbf{v}|_{1,\Omega} \le \bar{C}\, \|\mathbf{v}\|_{2,\Omega} \qquad \forall\ \mathbf{v} \in H^2(\Omega)^2 \tag{28}$$

where $\bar{C}$ depends only on Ω and σ from (7).

Now consider any $\overline{q} \in \bar{Q}_h$. We denote

$$q_1 = \frac{1}{\mathrm{meas}_2(\Omega)} \int_\Omega \overline{q}\,\mathrm{d}x \qquad\qquad q_2 = \overline{q} - q_1 \tag{29}$$

Then $q_2 \in \bar{Q}_h \cap L^2_0(\Omega)$. Setting $\mathbf{v}_1 = q_1\, \mathrm{meas}_2(\Omega)\, r_h \mathbf{u}$, we have $\mathbf{v}_1 \in \mathbf{R}_h$ and, applying the Gauss integral theorem, we infer that

$$(\mathrm{div}\, \mathbf{v}_1, q_1) = \|q_1\|^2_{0,\Omega} \qquad\qquad |\mathbf{v}_1|_{1,\Omega} \le \bar{C}\, \sqrt{\mathrm{meas}_2(\Omega)}\, \|\mathbf{u}\|_{2,\Omega}\, \|q_1\|_{0,\Omega} \tag{30}$$

According to Girault et al. (1986), the inf–sup condition (15) holds for $V = \mathbf{R}_h \cap H^1_0(\Omega)^2$, $Q = \bar{Q}_h \cap L^2_0(\Omega)$ and $b(\cdot,\cdot) = (\mathrm{div}\,\cdot,\cdot)$ with $\bar{\beta} > 0$ depending only on Ω and σ. Hence, in view of Corollary 1, there exists $\mathbf{v}_2 \in \mathbf{R}_h \cap H^1_0(\Omega)^2$ satisfying

$$(\mathrm{div}\, \mathbf{v}_2, q_2) = \|q_2\|^2_{0,\Omega} \qquad\qquad |\mathbf{v}_2|_{1,\Omega} \le \frac{1}{\bar{\beta}}\, \|q_2\|_{0,\Omega} \tag{31}$$

Now the lemma easily follows from Lemma 2.

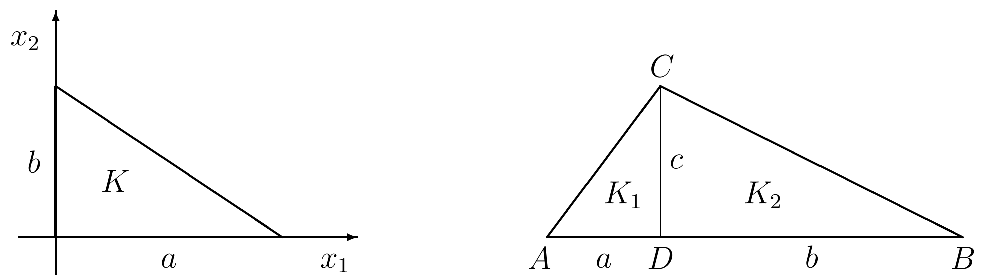

Figure 1. Notation for a right–angled triangle K (left) and a general triangle $K = K_1 \cup K_2$ (right).

Lemma 4 *For any $K \in \mathcal{T}_h^*$ and any $\widetilde{q} \in L^2_0(K)$, there exists $\widetilde{\mathbf{v}} \in H^1_0(K)^2$ such that*

$$(\mathrm{div}\, \widetilde{\mathbf{v}}, \widetilde{q})_K = \|\widetilde{q}\|^2_{0,K} \qquad\qquad |\widetilde{\mathbf{v}}|_{1,K} \le C\, \|\widetilde{q}\|_{0,K} \tag{32}$$

where C depends only on σ.

Proof. First, let us assume that K is a right–angled triangle. We choose a coordinate system such that the vertex of K opposite its hypotenuse lies in the origin and the legs of K lie on positive axes (cf. Fig. 1). Let a, b be the lengths of the edges of K lying on the axes x_1 and x_2, respectively. Defining a diagonal matrix $D = \mathrm{diag}\{a, b\}$, the mapping $x \mapsto D^{-1} x$ transforms the element K onto the reference triangle $\widehat{K}$ having the vertices $(0,0)$, $(1,0)$ and $(0,1)$.

Consider any $\widetilde{q} \in L^2_0(K)$ and set $\widehat{q}(\widehat{x}) = \widetilde{q}(D\,\widehat{x})$ for $\widehat{x} \in \widehat{K}$. Then $\widehat{q} \in L^2_0(\widehat{K})$. According to Girault et al. (1986), p. 81, the inf–sup condition (15) holds for $V = H^1_0(\widehat{K})^2$, $Q = L^2_0(\widehat{K})$ and $b(\cdot,\cdot) = (\mathrm{div}\,\cdot,\cdot)_{\widehat{K}}$ with some $\hat{\beta} > 0$, and hence it follows from Corollary 1 that there exists $\widehat{\mathbf{v}} \in H^1_0(\widehat{K})^2$ such that

$$(\mathrm{div}\, \widehat{\mathbf{v}}, \widehat{q})_{\widehat{K}} = \|\widehat{q}\|^2_{0,\widehat{K}} \qquad\qquad |\widehat{\mathbf{v}}|_{1,\widehat{K}} \le \frac{1}{\hat{\beta}}\, \|\widehat{q}\|_{0,\widehat{K}} \tag{33}$$

Setting $\widetilde{\mathbf{v}}(x) = D\,\widehat{\mathbf{v}}(D^{-1}\,x)$ for $x \in K$, we obtain a function from $H_0^1(K)^2$ satisfying

$$(\operatorname{div}\widetilde{\mathbf{v}}, \widetilde{q})_K = \|\widetilde{q}\|_{0,K}^2 \qquad\qquad |\widetilde{\mathbf{v}}|_{1,K} \le \frac{1}{\hat{\beta}} \max\left\{\frac{a}{b}, \frac{b}{a}\right\} \|\widetilde{q}\|_{0,K} \tag{34}$$

Since $a < h_K \le \sigma\,\varrho_K < \sigma\,b$ and, analogously, $b < \sigma\,a$, we see that (32) holds with $C = \sigma/\hat{\beta}$.

Now, let K be any element of $\mathcal{T}_h^*$ and let A, B, C be its vertices. We assume that the length of AB is h_K and we denote by D a point on AB such that CD is perpendicular to AB (cf. Fig. 1). Further, we denote by K_1 the triangle ADC, by K_2 the triangle DBC, and by a, b, c the lengths of AD, DB and CD, respectively. Then

$$a \ge h_K - \sqrt{h_K^2 - c^2} > \frac{c^2}{2\,h_K} > \frac{c\,\varrho_K}{2\,h_K} \ge \frac{c}{2\,\sigma} \tag{35}$$

Since $a/c < h_K/\varrho_K \le \sigma$, we deduce that

$$\max\left\{\frac{a}{c}, \frac{c}{a}\right\} < 2\,\sigma \qquad\qquad \max\left\{\frac{b}{c}, \frac{c}{b}\right\} < 2\,\sigma \tag{36}$$

Consider any $\widetilde{q} \in L_0^2(K)$ and let $q_1, q_2 \in L_0^2(K)$ be defined by

$$q_1|_{K_i} = \frac{1}{\operatorname{meas}_2(K_i)} \int_{K_i} \widetilde{q}\,\mathrm{d}x \qquad i = 1,2 \qquad\qquad q_2 = \widetilde{q} - q_1 \tag{37}$$

Then, denoting $M_i = \operatorname{meas}_2(K_i)$, $i = 1,2$, we have $q_1|_{K_1}\,M_1 + q_1|_{K_2}\,M_2 = 0$ and hence

$$\|q_1\|_{0,K} = \left|q_1|_{K_1}\right| \sqrt{\frac{M_1\,(M_1 + M_2)}{M_2}} = \left|q_1|_{K_1} - q_1|_{K_2}\right| \sqrt{\frac{M_1\,M_2}{M_1 + M_2}} \le \frac{1}{2}\,h_K \left|q_1|_{K_1} - q_1|_{K_2}\right| \tag{38}$$

Let $\varphi = \lambda_A\,\lambda_B\,\lambda_C$, where λ_A, λ_B, λ_C are the barycentric coordinates on K with respect to the vertices A, B, C, respectively. Then $\varphi \in H_0^1(K)$ and

$$\int_{CD} \varphi\,\mathrm{d}\sigma = \frac{a\,b\,c}{12\,h_K^2} > \frac{h_K}{48\,\sigma^3} \qquad\qquad |\varphi|_{1,K} \le \bar{C} \tag{39}$$

where $\bar{C}$ depends only on σ. Let $\mathbf{n}_{CD}$ be the unit normal vector to CD pointed into K_2 and let us set

$$\mathbf{v}_1 = \alpha\,\varphi\,\mathbf{n}_{CD} \qquad \text{with} \qquad \alpha = \frac{\|q_1\|_{0,K}^2}{\int_{CD} \varphi\,\mathrm{d}\sigma\,(q_1|_{K_1} - q_1|_{K_2})} \qquad (\alpha = 0 \ \text{ if } \ q_1 = 0) \tag{40}$$

Using (38) and (39), we obtain $|\alpha| \le 24\,\sigma^3\,\|q_1\|_{0,K}$ and hence, applying the Gauss integral theorem, we infer that

$$(\operatorname{div}\mathbf{v}_1, q_1)_K = \|q_1\|_{0,K}^2 \qquad\qquad |\mathbf{v}_1|_{1,K} \le 24\,\sigma^3\,\bar{C}\,\|q_1\|_{0,K} \tag{41}$$

Further, since $q_2|_{K_i} \in L_0^2(K_i)$, $i = 1,2$, it follows from the first part of the proof and from (36) that there exists $\mathbf{v}_2 \in H_0^1(K)^2$ such that $\mathbf{v}_2 = \mathbf{0}$ on CD and

$$(\operatorname{div}\mathbf{v}_2, q_2)_K = \|q_2\|_{0,K}^2 \qquad\qquad |\mathbf{v}_2|_{1,K} \le \frac{2\,\sigma}{\hat{\beta}}\,\|q_2\|_{0,K} \tag{42}$$

Applying Lemma 2, we obtain (32).

Now we are in a position to prove the validity of the inf–sup condition (10) for $Q = L^2(\Omega)$.

Theorem 1 *There exists a constant $\beta > 0$ depending only on Ω, Γ^D and σ such that*

$$\sup_{\mathbf{v}\in\mathbf{Z}_h\setminus\{\mathbf{0}\}} \frac{(\operatorname{div}\mathbf{v}, q)}{|\mathbf{v}|_{1,\Omega}} \geq \beta\, \|q\|_{0,\Omega} \qquad \forall\, q \in L^2(\Omega) \tag{43}$$

where the space $\mathbf{Z}_h$ was defined at the beginning of this section.

Proof. Let us consider any $q \in L^2(\Omega)$ and denote by q_1, q_2 functions defined by

$$q_1|_K = \frac{1}{\operatorname{meas}_2(K)} \int_K q\,\mathrm{d}x \qquad \forall\, K \in \mathcal{T}_h^* \qquad\qquad q_2 = q - q_1 \tag{44}$$

Then $q_1 \in \bar{Q}_h$ and $q_2|_K \in L^2_0(K)$ for any $K \in \mathcal{T}_h^*$. According to Lemmas 3 and 4, there exist functions $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{Z}_h$ such that $\mathbf{v}_2|_K \in H^1_0(K)^2$ for any $K \in \mathcal{T}_h^*$ and

$$(\operatorname{div}\mathbf{v}_i, q_i) = \|q_i\|^2_{0,\Omega} \qquad\qquad |\mathbf{v}_i|_{1,\Omega} \leq C\,\|q_i\|_{0,\Omega} \qquad\qquad i = 1, 2 \tag{45}$$

where C depends only on Ω, Γ^D and σ. Thus, the theorem follows from Lemma 2.

4 Discrete Korn's Inequality for Spaces Satisfying the Patch Test of Order 2

Any nonconforming finite element space which is defined over the triangulation $\mathcal{T}_h$, approximates the space $\mathbf{V}$ and satisfies the patch test of order 2 is a subspace of the space

$$\mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^2 \,:\, \mathbf{v}|_K \in H^1(K)^2 \;\; \forall\, K \in \mathcal{T}_h\,, \\ \int_E [|\mathbf{v}|]_E\, q\,\mathrm{d}\sigma = \mathbf{0} \;\; \forall\, q \in P_1(E),\, E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D\} \tag{46}$$

In this section we shall show that a discrete analogue of the Korn inequality (6) holds for this space with a constant C independent of h. The basic idea of the proof is the same as in Falk (1991) where the discrete Korn inequality was established for some particular subspaces of $\mathbf{V}_h$.

Following Falk (1991), we introduce the notation

$$\operatorname{rot}\mathbf{v} = -\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \qquad\qquad \operatorname{curl}\mathbf{z} = \begin{pmatrix} \dfrac{\partial z_1}{\partial x_2} & -\dfrac{\partial z_1}{\partial x_1} \\[2mm] \dfrac{\partial z_2}{\partial x_2} & -\dfrac{\partial z_2}{\partial x_1} \end{pmatrix} \tag{47}$$

First let us prove the following orthogonality result.

Lemma 5 *Let $\mathbf{Z}_h$ be the space defined at the beginning of the previous section. Then*

$$\sum_{K\in\mathcal{T}_h} (\nabla\mathbf{v}, \operatorname{curl}\mathbf{z})_K = 0 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h,\, \mathbf{z} \in \mathbf{Z}_h \tag{48}$$

Proof. For any element K, let $\mathbf{n}_{\partial K} = (n_{\partial K1}, n_{\partial K2})$ be the unit outer normal vector to the boundary of K. We denote by $\mathbf{t}_{\partial K} = (-n_{\partial K2}, n_{\partial K1})$ the corresponding tangent vector. Applying the Gauss integral theorem, we obtain

$$(\nabla\mathbf{v}, \operatorname{curl}\mathbf{z})_K = \int_{\partial K} v_1 \frac{\partial z_1}{\partial \mathbf{t}_{\partial K}} + v_2 \frac{\partial z_2}{\partial \mathbf{t}_{\partial K}}\,\mathrm{d}\sigma \qquad \forall\, \mathbf{v} \in H^1(K)^2,\, \mathbf{z} \in H^2(K)^2 \tag{49}$$

For $\mathbf{z} \in C_0^\infty(K)^2$, the right–hand side of (49) vanishes and since the space $C_0^\infty(K)$ is dense in $H^1_0(K)$, we infer that

$$(\nabla\mathbf{v}, \operatorname{curl}\mathbf{z})_K = 0 \qquad \forall\, \mathbf{v} \in H^1(K)^2,\, \mathbf{z} \in H^1_0(K)^2 \tag{50}$$

If $\mathbf{z} \in \mathbf{Z}_h$, then $\mathbf{z} = \mathbf{r} + \mathbf{g}$ with $\mathbf{r} \in \mathbf{R}_h$ and $\mathbf{g} \in \mathbf{G}_h$. Thus, it follows from (50) that, for any $\mathbf{v} \in \mathbf{V}_h$,

$$\sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v}, \operatorname{curl} \mathbf{z})_K = \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v}, \operatorname{curl} \mathbf{r})_K \tag{51}$$

The derivatives along edges of functions from $\mathbf{R}_h$ are continuous across edges and vanish on edges from $\mathcal{E}_h^N$ and hence we deduce using (49) that

$$\sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v}, \operatorname{curl} \mathbf{z})_K = \sum_{E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D} \int_E [|v_1|]_E \frac{\partial r_1}{\partial \mathbf{t}_E} + [|v_2|]_E \frac{\partial r_2}{\partial \mathbf{t}_E} \, \mathrm{d}\sigma \tag{52}$$

Since $\partial r_1 / \partial \mathbf{t}_E$, $\partial r_2 / \partial \mathbf{t}_E \in P_1(E)$ for any $E \in \mathcal{E}_h$, the right–hand side of (52) vanishes due to the definition of the space $\mathbf{V}_h$.

Now we have prepared all tools for proving a Korn inequality for the above space $\mathbf{V}_h$.

Theorem 2 *There exists a constant C depending only on Ω, Γ^D and σ from (7) such that*

$$\sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2 \le C \sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} + \nabla \mathbf{v}^\mathsf{T}\|_{0,K}^2 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h \tag{53}$$

Proof. Consider any $\mathbf{v} \in \mathbf{V}_h$. According to Theorem 1 and Lemma 1, there exists $\mathbf{z} \in \mathbf{Z}_h$ such that

$$(\operatorname{div} \mathbf{z}, q) = \sum_{K \in \mathcal{T}_h} (\operatorname{rot} \mathbf{v}, q)_K \qquad \forall\, q \in L^2(\Omega) \qquad\qquad |\mathbf{z}|_{1,\Omega} \le \frac{\sqrt{2}}{\beta} \sqrt{\sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2} \tag{54}$$

Particularly, setting $q|_K = \operatorname{rot}(\mathbf{v}|_K)$ for any $K \in \mathcal{T}_h$, we get

$$\sum_{K \in \mathcal{T}_h} (\operatorname{rot} \mathbf{v} - \operatorname{div} \mathbf{z}, \operatorname{rot} \mathbf{v})_K = 0 \tag{55}$$

It can be readily verified (cf. also Falk, 1991) that, on each element $K \in \mathcal{T}_h$, we have

$$|\nabla \mathbf{v}|^2 = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^\mathsf{T}) \cdot (\nabla \mathbf{v} - \operatorname{curl} \mathbf{z}) + \nabla \mathbf{v} \cdot \operatorname{curl} \mathbf{z} + \frac{1}{2} (\operatorname{rot} \mathbf{v} - \operatorname{div} \mathbf{z}) \operatorname{rot} \mathbf{v} \tag{56}$$

Thus, applying (48) and (55), we infer that

$$2 \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2 = \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v} + \nabla \mathbf{v}^\mathsf{T}, \nabla \mathbf{v} - \operatorname{curl} \mathbf{z})_K \tag{57}$$

$$\le \sqrt{\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} + \nabla \mathbf{v}^\mathsf{T}\|_{0,K}^2} \sqrt{\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} - \operatorname{curl} \mathbf{z}\|_{0,K}^2} \tag{58}$$

In view of (54), we have

$$\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} - \operatorname{curl} \mathbf{z}\|_{0,K}^2 \le \frac{2\,\beta^2 + 4}{\beta^2} \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2 \tag{59}$$

and the theorem follows.

Theorem 2 implies the uniform validity of the discrete Korn inequality for many known nonconforming finite elements, e.g., for the nonconforming quadratic element of Fortin et al. (1983), for the nonconforming cubic element of Crouzeix et al. (1989) or for the P_1^{mod} element of Knobloch et al. (1999). Note also that Theorem 2 implies the validity of the Korn inequality (6) since $\mathbf{V} \subset \mathbf{V}_h$.

5 Counterexamples for First Order Finite Element Spaces

In this section we show that Theorem 2 does not hold for typical nonconforming first order finite element spaces. This is well known for the linear triangular Crouzeix–Raviart element introduced in Crouzeix et al. (1973) for which we can even find a triangulation $\mathcal{T}_h$ such that the right–hand side of (53) vanishes for some $\mathbf{v} \in \mathbf{V}_h$ (cf. Falk et al., 1990). Therefore, we restrict ourselves to the quadrilateral case. The triangular case can be investigated analogously. The proof follows ideas of Falk et al. (1990).

Let $\mathcal{T}_h$ be a triangulation of $\Omega = (0,1)^2$ consisting of equal squares K with $h_K = h$. We denote by $\widehat{K} = (0,1)^2$ the reference element and by $\widehat{Q}$ a space of functions on $\widehat{K}$ satisfying $P_1(\widehat{K}) \subset \widehat{Q} \subset H^2(\widehat{K})$ and $\dim \widehat{Q} = 4$. We introduce functionals $\widehat{I}_i$, $i = 1, \dots, 4$, and I_E, J_E, $E \in \mathcal{E}_h$, defined either by

$$\widehat{I}_i(\widehat{v}) = \int_{\widehat{E}_i} \widehat{v}\, \mathrm{d}\widehat{\sigma} \qquad I_E(v) = \frac{1}{\mathrm{meas}_1(E)} \int_E \langle|v|\rangle_E \, \mathrm{d}\sigma \qquad J_E(v) = \int_E [|v|]_E \, \mathrm{d}\sigma \tag{60}$$

where $\widehat{E}_1, \dots, \widehat{E}_4$ are edges of $\widehat{K}$, or by

$$\widehat{I}_i(\widehat{v}) = \widehat{v}(\widehat{C}_i) \qquad I_E(v) = \langle|v|\rangle_E(C_E) \qquad J_E(v) = [|v|]_E(C_E) \tag{61}$$

where $\widehat{C}_i$ and C_E are the midpoints of $\widehat{E}_i$ and E, respectively. We assume that the functionals $\widehat{I}_1, \dots, \widehat{I}_4$ are unisolvent with $\widehat{Q}$ and we introduce a general nonconforming first order space

$$\mathbf{W}_h = \{\mathbf{v} \in L^2(\Omega)^2 \; : \; \mathbf{v} \circ F_K \in [\widehat{Q}]^2 \;\; \forall\, K \in \mathcal{T}_h \,, \; J_E(\mathbf{v}) = \mathbf{0} \;\; \forall\, E \in \mathcal{E}_h\} \tag{62}$$

where $F_K : \widehat{K} \to K$ is any regular affine mapping which maps $\widehat{K}$ onto K. The space $\mathbf{W}_h$ defined using the functionals (60) consists of functions satisfying the patch test of order 1. This is generally not true for $\mathbf{W}_h$ defined using the functionals (61) which consists of functions continuous in the midpoints of edges. It is easy to show that, for both types of spaces and for any fixed h, the root of the right–hand side of (53) is a norm on $\mathbf{W}_h$ and hence the inequality (53) holds for any $\mathbf{v} \in \mathbf{W}_h$ with some constant C_h. However, this constant C_h tends to infinity for $h \to 0$ as we prove in the following theorem.

Theorem 3 *For any $h < 1/3$, the space $\mathbf{W}_h$ satisfies*

$$\sup_{\mathbf{W}_h \setminus \{\mathbf{0}\}} \frac{\sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2}{\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\|_{0,K}^2} \geq \frac{1}{6\, M^2\, h} \tag{63}$$

where $M = \sup\{|\widehat{q}|_{1,\widehat{K}} \; : \; \widehat{q} \in \widehat{Q}\,, \;\; |\widehat{I}_i(\widehat{q})| \leq 1, \; i = 1, \dots, 4\}$.

Proof. Let $\mathcal{T}_h^1 \subset \mathcal{T}_h$ consist of elements whose closures intersect $\partial\Omega$ and let $\mathcal{T}_h^2$, $\mathcal{T}_h^3$ divide $\mathcal{T}_h \setminus \mathcal{T}_h^1$ in a checkerboard manner. For any $K \in \mathcal{T}_h$, let $\mathbf{v}_K(x) = \frac{1}{h}\,(x_2 - x_{K2}, x_{K1} - x_1)$, where (x_{K1}, x_{K2}) is the barycentre of K. We introduce a function $\mathbf{v} \in \mathbf{W}_h$ uniquely determined by

$$\mathbf{v}|_K = \mathbf{v}_K \quad \forall\, K \in \mathcal{T}_h^2 \qquad \mathbf{v}|_K = -\mathbf{v}_K \quad \forall\, K \in \mathcal{T}_h^3 \qquad I_E(\mathbf{v}) = 0 \quad \forall\, E \in \mathcal{E}_h^1 \tag{64}$$

where $\mathcal{E}_h^1 \subset \mathcal{E}_h^i$ consists of edges which do not belong to elements from $\mathcal{T}_h \setminus \mathcal{T}_h^1$. Since $\nabla \mathbf{v}_K + \nabla \mathbf{v}_K^{\mathsf{T}} = 0$, we have

$$\sum_{K \in \mathcal{T}_h} \|\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\|_{0,K}^2 = \sum_{K \in \mathcal{T}_h^1} \|\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathsf{T}}\|_{0,K}^2 \leq 4 \sum_{K \in \mathcal{T}_h^1} |\mathbf{v}|_{1,K}^2 = 4 \sum_{K \in \mathcal{T}_h^1} |\mathbf{v} \circ F_K|_{1,\widehat{K}}^2 \leq \frac{3\, M^2}{h} \tag{65}$$

Further, we have $|\mathbf{v}_K|_{1,K}^2 = 1$ and hence

$$\sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2 \geq \sum_{K \in \mathcal{T}_h^2 \cup \mathcal{T}_h^3} |\mathbf{v}|_{1,K}^2 = \frac{(\sqrt{2} - 2\, h)^2}{h^2} \geq \frac{1}{2\, h^2} \tag{66}$$

which gives (63).

The above result shows that Theorem 2 does not hold for quadrilateral elements of Rannacher et al. (1992) and Cai et al. (1999) and for their various modifications which can be found in the literature. Let us remark that the results of the present paper do not cover all possible nonconforming finite element spaces. For example, in Kouhia et al. (1993), a linear triangular finite element being nonconforming in one component only was introduced and it was shown that it satisfies the discrete Korn inequality.

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