MODIFIED FE DISCRETIZATIONS OF INCOMPRESSIBLE FLOW PROBLEMS AND THEIR RELATIONSHIP TO STABILIZED METHODS

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We consider conforming finite element discretizations of two-dimensional stationary incompressible Stokes and Navier–Stokes equations based on triangular and quadrilateral meshes. We present a general approach for interpreting stabilized methods as modified Galerkin discretizations employing bubble functions. This interpretation provides a deeper insight into the behaviour of stabilized methods and is a valuable tool for their theoretical investigations. Further, we show how new types of stabilizations can be derived.

1 Introduction

We consider the Stokes equations

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \tag{1}$$

and the Navier–Stokes equations

$$-\nu\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \tag{2}$$

defined in a bounded domain $\Omega \subset \mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. The equations describe the velocity $\boldsymbol{u}$ and the pressure p in an incompressible linear viscous fluid. The kinematic viscosity $\nu > 0$ is assumed to be constant and $\boldsymbol{f}$ denotes an external body force, e.g., the gravity. Denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x, \qquad n(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) = \int_\Omega \boldsymbol{v}\cdot(\nabla\boldsymbol{w})\boldsymbol{u}\,\mathrm{d}x,$$

$$b(\boldsymbol{v},p) = -\int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x,$$

the usual weak formulation of (1) reads: Given $\nu > 0$ and $\boldsymbol{f} \in H^{-1}(\Omega)^2$, find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\nu\, a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^2,\ q \in L_0^2(\Omega). \tag{3}$$

It can be shown that the problem (3) has a unique solution (cf. [8], p. 80, Theorem 5.1). The weak formulation of (2) is given by

$$\nu\, a(\boldsymbol{u}, \boldsymbol{v}) + n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^2,\ q \in L_0^2(\Omega).$$

This problem has a solution which is unique if ν is sufficiently large and/or $\boldsymbol{f}$ is sufficiently small (cf. [8], pp. 291 and 292).

Replacing the spaces $H_0^1(\Omega)^2$ and $L_0^2(\Omega)$ in (3) by some finite element subspaces V_h and Q_h, respectively, we obtain a conforming (or Galerkin) finite element discretization of (1): Find $\boldsymbol{u}_h \in \mathrm{V}_h$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\nu\, a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) - b(\boldsymbol{u}_h, q_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h,\ q_h \in \mathrm{Q}_h. \tag{4}$$

A conforming discretization of (2) can be obtained analogously.

Conforming discretizations of (1) and (2) can suffer from various instabilities. The two main origins are unstable pairs of spaces and dominated convection. The spaces V_h, Q_h are unstable if the Babuška–Brezzi condition

$$\exists \beta > 0 : \quad \sup_{\boldsymbol{v}_h \in \mathrm{V}_h \setminus \{0\}} \frac{b(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,\Omega}} \geq \beta \|q_h\|_{0,\Omega} \quad \forall q_h \in \mathrm{Q}_h,\ h > 0, \tag{5}$$

does not hold. A consequence are spurious pressure oscillations and the discrete problem may even have not any solution. If the convective term in the Navier–Stokes equations dominates the diffusion term, inner and boundary layers occur which cannot be resolved unless the discretization parameter h is very small. That often leads to spurious velocity oscillations.

During the past two decades, many various stabilizations have been developed in which the unstable behaviour is mostly suppressed by adding suitable mesh-dependent terms to the conforming discretization. Another possibility to remove instabilities due to unstable pairs of spaces is to enlarge the velocity space in such a way that the Babuška–Brezzi condition is fulfilled. For this, special supplementary functions have been designed which assure the stability and lead to sufficiently small velocity spaces. Such supplementary functions are often known as bubble functions or, shortly, bubbles.

It was realized very soon [14] that there is a close relationship between stabilized methods and Galerkin methods with bubble functions. Namely, if we drop the bubble part of the solution of a Galerkin method with bubble functions, then we sometimes get functions which represent a solution of a stabilized method. For linear problems, this relationship was also established in an abstract framework via virtual bubbles in [2]. There is a lot of further

papers devoted to investigations of the mentioned relationship (see [12] for references), but most of them only deal with linear problems, triangular elements, linear ansatz functions for the space V_h and usual bubble functions which have their supports always in one element only.

In this paper, we consider a more general situation, particularly, the fully nonlinear Navier–Stokes equations, both triangular and quadrilateral elements and rather general bubble-like functions. We show that, in addition to adding bubble functions, a modification of some terms in the conforming discretization is typically necessary for recovering stabilized methods and we develop a general methodology for recovering and deriving stabilized discretizations of incompressible flow problems.

The paper is organized in the following way. In Section 2 we give examples of bubble functions. In Section 3 we introduce a general class of modified discretizations of the Stokes equations which is then used in Section 4 to derive and recover various stabilized methods. Finally, Section 5 is devoted to deriving of bubble-based stabilizations of the Navier–Stokes equations. Throughout the paper we use a standard notation which can be found in [8].

2 Examples of Finite Element Spaces

We consider usual regular triangulations $\mathcal{T}_h$ of Ω consisting of closed triangles or closed convex quadrilaterals T and we denote

$$\mathrm{V}_h^1 = \{v \in H_0^1(\Omega)^2;\, v|_T \in R(T)^2 \quad \forall T \in \mathcal{T}_h\}, \tag{6}$$

where $R(T) = P_1(T)$ if $\mathcal{T}_h$ consists of triangles. If $\mathcal{T}_h$ consists of quadrilaterals, then $R(T)$ is the space of functions on T which are bilinear on the reference element. The space Q_h is defined either by

$$\mathrm{Q}_h = \{q \in L_0^2(\Omega);\, q|_T \in P_0(T) \quad \forall T \in \mathcal{T}_h\} \tag{7}$$

or by

$$\mathrm{Q}_h = \{q \in C(\overline{\Omega}) \cap L_0^2(\Omega);\, q|_T \in R(T) \quad \forall T \in \mathcal{T}_h\}. \tag{8}$$

Thus, we have four pairs of spaces V_h^1, Q_h, but none of them satisfies the Babuška–Brezzi condition (5). Therefore, in each case, we enrich the space V_h^1 by a suitable space V_h^2 so that (5) holds for the spaces $\mathrm{V}_h = \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$, Q_h.

If Q_h is defined by (7), then we can set $\mathrm{V}_h^2 = \operatorname{span}\{\varphi_E \boldsymbol{n}_E\}_{E \not\subset \partial\Omega}$, where $\varphi_E \in H_0^1(\Omega)$ are finite element functions assigned to inner edges E of the triangulation which have their supports in the two elements adjacent to E and satisfy $\int_E \varphi_E \,\mathrm{d}\sigma \neq 0$. The vectors $\boldsymbol{n}_E$ are fixed normal vectors to the edges. We refer to [3] and [6] for particular examples of φ_E.

If Q_h is defined by (8) and $\mathcal{T}_h$ consists of triangles, we can set $\mathrm{V}_h^2 = [\mathrm{span}\,\{\varphi_T\}_{T\in\mathcal{T}_h}]^2$, where each $\varphi_T \in H_0^1(\Omega)$ is a finite element function having its support in the element T. Thus, the space V_h^1 is enriched by two vector bubbles on each element. For example, φ_T can be the product of the barycentric coordinates on T, which gives the so–called mini element [1]. If $\mathcal{T}_h$ consists of quadrilaterals, the stability is assured by four bubbles on each element [13].

The spaces V_h^2 from the previous paragraph are particular examples of the general space $\mathrm{V}_h^2 = \mathrm{span}\,\{\varphi_h^i\,\boldsymbol{t}_h^i\}_{i=1}^{N_h}$, where $\boldsymbol{t}_h^i$ are unit vectors and $\varphi_h^i \in H_0^1(\Omega)$ are finite element functions having their supports in one element or in two elements possessing a common edge. To distinguish which element or elements a function φ_h^i belongs to, we introduce points A_h^i different from the vertices of $\mathcal{T}_h$. If A_h^i lies in the interior of some element T, we require that $\mathrm{supp}\,\varphi_h^i \subset T$ and if A_h^i lies on an edge E, we require that $\mathrm{supp}\,\varphi_h^i$ lies in the two elements adjacent to E and that $\boldsymbol{t}_h^i \| E$. For triangular meshes, we assume that there exist two points $A_h^i \in T$ for each element T. In the quadrilateral case, three points $A_h^i \in T$ are supposed for any T. In other words, we need two functions φ_h^i per element in the triangular case and three functions φ_h^i per element in the quadrilateral case. In both cases, each function may be common to two elements. Under some further assumptions [12] which are satisfied for the spaces used in Sections 4 and 5, it can be shown that the space V_h^2 assures the validity of (5) for Q_h defined by (8). Particularly, we can use spaces of the type $\mathrm{V}_h^2 = \mathrm{span}\,\{\varphi_E\,\boldsymbol{t}_E\}_E$, where $\boldsymbol{t}_E$ are tangential vectors to the edges.

3 Modified Discretizations of the Stokes Equations

Let us consider a general stabilized method for the Stokes problem: Find $\boldsymbol{u}_h^1 \in \mathrm{V}_h^1$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h^1\rangle \quad \forall \boldsymbol{v}_h^1 \in \mathrm{V}_h^1, \tag{9}$$

$$b(\boldsymbol{u}_h^1, q_h) + c_h(\boldsymbol{u}_h^1, p_h, q_h) = 0 \qquad \forall q_h \in \mathrm{Q}_h, \tag{10}$$

where $c_h(\boldsymbol{u}_h^1, p_h, q_h)$ is a mesh-dependent term compensating the instability of some finite element spaces V_h^1, Q_h. We would like to recover this method using the conforming discretization (4) employing bubble functions from a space V_h^2 assuring the validity of (5). The discretization (4) can be written in the form: Find $\boldsymbol{u}_h \in \mathrm{V}_h \equiv \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and $p_h \in \mathrm{Q}_h$ such that

$$\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + \nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h^1\rangle \quad \forall \boldsymbol{v}_h^1 \in \mathrm{V}_h^1, \tag{11}$$

$$\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^2) + \nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^2) + b(\boldsymbol{v}_h^2, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h^2\rangle \quad \forall \boldsymbol{v}_h^2 \in \mathrm{V}_h^2, \tag{12}$$

$$b(\boldsymbol{u}_h^1, q_h) + b(\boldsymbol{u}_h^2, q_h) = 0 \qquad \forall q_h \in \mathrm{Q}_h, \tag{13}$$

where $\boldsymbol{u}_h^1$ is the part of $\boldsymbol{u}_h$ lying in V_h^1 and $\boldsymbol{u}_h^2$ is the bubble part of $\boldsymbol{u}_h$ lying in V_h^2. If we use the mini-element, then $a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^1) = a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^2) = 0$ and hence the equations (11) and (9) are identical. Therefore, we have a chance to recover the stabilized method (9)–(10), i.e., to show that $b(\boldsymbol{u}_h^2, q_h) = c_h(\boldsymbol{u}_h^1, p_h, q_h)$. If, however, we use more general bubble-like functions, for example those ones from Section 2 used to stabilize pairs of spaces with Q_h defined by (7), then the term $a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^1)$ generally does not vanish and a direct equivalence with a stabilized method of the type (9)–(10) cannot be shown.

Therefore, we come to the idea to drop the term $\nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^1)$ from the conforming discretization. Although this may seem a little bit suspicious, we again obtain a reasonable discretization. We can even consider the following more general modified problem: Find $\boldsymbol{u}_h \in \mathrm{V}_h$ and $p_h \in \mathrm{Q}_h$ such that

$$\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + \alpha_1\, \nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle \quad \forall \boldsymbol{v}_h^1 \in \mathrm{V}_h^1, \tag{14}$$

$$\alpha_2\, \nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^2) + \alpha_3\, \nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^2) + b(\boldsymbol{v}_h^2, p_h) = \langle \boldsymbol{g}, \boldsymbol{v}_h^2 \rangle \quad \forall \boldsymbol{v}_h^2 \in \mathrm{V}_h^2, \tag{15}$$

$$b(\boldsymbol{u}_h^1, q_h) + b(\boldsymbol{u}_h^2, q_h) = 0 \qquad \forall q_h \in \mathrm{Q}_h, \tag{16}$$

where $\alpha_3 > 0$, $|\alpha_1 + \alpha_2| \le 2\sqrt{\alpha_3}$ and $\boldsymbol{g} = \boldsymbol{f}$ or $\boldsymbol{g} = \boldsymbol{0}$. If $\boldsymbol{g} = \boldsymbol{0}$, we assume that $\boldsymbol{f} \in L^2(\Omega)^2$. It was shown in [11] that the modified problem (14)–(16) is uniquely solvable and that

$$|\boldsymbol{u} - \boldsymbol{u}_h^1|_{1,\Omega} \to 0, \quad |\boldsymbol{u}_h^2|_{1,\Omega} \to 0, \quad \|p - p_h\|_{0,\Omega} \to 0 \qquad \text{for } h \to 0, \tag{17}$$

where $\boldsymbol{u}$, p is the weak solution of the Stokes problem (1). If $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$, then the convergence in (17) is linear and if $\alpha_2 = 1$, $\boldsymbol{g} = \boldsymbol{f}$ and $\boldsymbol{u}$, p have a higher regularity, then we obtain the usual optimal convergence rates corresponding to the spaces V_h^1 and Q_h.

4 Deriving of Stabilized Methods

In this section, we give examples of stabilized methods which can be derived by eliminating the space V_h^2 from a modified discretization (14)–(16).

First, let us consider the case $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, $\boldsymbol{g} = \boldsymbol{0}$. Let V_h^1, Q_h be given by (6), (7), respectively, and let $\mathrm{V}_h^2 = \operatorname{span}\{\varphi_E\, \boldsymbol{n}_E\}_{E \not\subset \partial\Omega}$ (see Section 2). Defining φ_E in such a way that $a(\varphi_E\, \boldsymbol{n}_E, \varphi_{E'}\, \boldsymbol{n}_{E'}) = 0$ for any edges $E \ne E'$, we can compute $\boldsymbol{u}_h^2$ from (15) and substitute it in (16), which gives

$$b(\boldsymbol{u}_h^1, q_h) - \sum_{E \not\subset \partial\Omega} \frac{\delta_E\, h_E}{\nu} \int_E [|p_h|]_E\, [|q_h|]_E \,\mathrm{d}\sigma = 0, \quad \delta_E = \frac{|\int_E \varphi_E \,\mathrm{d}\sigma|^2}{h_E^2 |\varphi_E|_{1,\Omega}^2},$$

where $[|p_h|]_E$ is the jump of p_h across E. This is a particular case of a stabilization by Hughes and Franca [9]. Proceeding analogously for V_h^1, Q_h

defined by (6), (8), respectively, and for the space $\mathrm{V}_h^2 = \operatorname{span}\{\varphi_E \boldsymbol{t}_E\}_E$ from Section 2 satisfying $a(\varphi_E \boldsymbol{t}_E, \varphi_{E'} \boldsymbol{t}_{E'}) = 0$ for $E \neq E'$, we get

$$b(\boldsymbol{u}_h^1, q_h) - \sum_E \frac{\delta_E h_E^3}{\nu} \int_E \frac{\partial p_h}{\partial \boldsymbol{t}_E} \frac{\partial q_h}{\partial \boldsymbol{t}_E} \,\mathrm{d}\sigma = 0, \quad \delta_E = \frac{|\int_\Omega \varphi_E \,\mathrm{d}x|^2}{h_E^4 |\varphi_E|_{1,\Omega}^2},$$

which is a new type of stabilization. Let us still consider the spaces V_h^1, Q_h defined by (6), (8), respectively, but together with a space V_h^2 of the type $\mathrm{V}_h^2 = \operatorname{span}\{\varphi_h^i \boldsymbol{t}_h^i\}_{i=1}^{N_h}$. In the triangular case, we take the mini-element and in the quadrilateral case, we assume that the elements of $\mathcal{T}_h$ are rectangles and, for any element T, we use four functions φ_h^i with disjoint supports in T. The corresponding vectors $\boldsymbol{t}_h^i$ are parallel to the edges of T, see [12] for details. Then we recover the stabilization by Brezzi and Pitkäranta [4]

$$b(\boldsymbol{u}_h^1, q_h) - \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T \nabla p_h \cdot \nabla q_h \,\mathrm{d}x = 0,$$

where γ_T is bounded from below and from above by positive constants independent of h and ν. If $\boldsymbol{g} = \boldsymbol{f}$ in (15) and $\boldsymbol{f}$ is piecewise linear, resp. bilinear, then we obtain the stabilization

$$b(\boldsymbol{u}_h^1, q_h) - \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T (\nabla p_h - \boldsymbol{f}) \cdot \nabla q_h \,\mathrm{d}x = 0$$

which is identical with the Petrov–Galerkin formulation of the Stokes equations introduced in [10]. Of course, for the mini-element, the two last results are classical, but in the quadrilateral case these results are nontrivial.

Finally, let us consider spaces V_h^1, Q_h consisting of continuous piecewise quadratic functions on triangles. These spaces do not satisfy (5). Dividing any element T into four equal triangles by connecting midpoints of edges and introducing two vector bubble functions from the mini-element on each subtriangle having a common vertex with T, we obtain a space V_h^2 assuring the stability. If we eliminate this space V_h^2 from the original conforming discretization (11)–(13) and assume that $\boldsymbol{f}$ is piecewise linear, we obtain the stabilization by Hughes and Franca [9]

$$\begin{aligned} &\nu a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) - b(\boldsymbol{u}_h^1, q_h) + \\ &\quad + \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T (-\nu \Delta \boldsymbol{u}_h^1 + \nabla p_h - \boldsymbol{f}) \cdot (\nu \Delta \boldsymbol{v}_h^1 + \nabla q_h) \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle. \end{aligned}$$

However, according to Section 3, we can also use the discretization (14)–(16) with $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$ and $\boldsymbol{g} = \boldsymbol{f}$ which guarantees the same asymptotic convergence rates of the discrete solution as (11)–(13). Then, eliminating the

space V_h^2, we obtain the absolutely stabilized method by Douglas and Wang [5] which differs from the above method by the sign in front of $\nu\Delta \boldsymbol{v}_h^1$.

5 Modified Discretizations of the Navier–Stokes Equations

In case of the Navier–Stokes equations, not only the bilinear form a and the right-hand side $\langle \boldsymbol{f}, \boldsymbol{v}_h^2 \rangle$, but also the convective term $n(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$ can be modified. Using the decomposition $\boldsymbol{u}_h = \boldsymbol{u}_h^1 + \boldsymbol{u}_h^2$, $\boldsymbol{v}_h = \boldsymbol{v}_h^1 + \boldsymbol{v}_h^2$, this term splits in a number of terms which can all be dropped or modified except for the term $n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h^1)$. The dropping of some parts of the convective term is essential since otherwise the function $\boldsymbol{u}_h^2$ generally cannot be eliminated from the discretization. For example, we can set

$$n(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) \approx n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h) + n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^2, \boldsymbol{v}_h^1) \approx n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h) - n(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1, \boldsymbol{u}_h^2).$$

The latter modification is motivated by the fact that $n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^2, \boldsymbol{v}_h^1) = -n(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1, \boldsymbol{u}_h^2) - \int_\Omega \boldsymbol{u}_h^2 \cdot \boldsymbol{v}_h^1 \operatorname{div} \boldsymbol{u}_h^1 \, \mathrm{d}x$ and $\boldsymbol{u}_h^1$ approximates a divergence-free function. Modifying the bilinear form a and the right-hand side as for the Stokes problem, we can prove [11] for a large class of modifications of $n(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$ that, for any nonsingular weak solution $\boldsymbol{u}$, p of (2) and sufficiently small h, the modified discrete problem is uniquely solvable in a neighbourhood of $(\boldsymbol{u}, p)$ and (17) holds. If $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$, then the convergence in (17) is linear and if $\alpha_2 = 1$, $\boldsymbol{g} = \boldsymbol{f}$, the term $n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h^2)$ is not modified and $\boldsymbol{u}$, p have a higher regularity, then we obtain the usual optimal convergence rates corresponding to the spaces V_h^1 and Q_h.

As an example, let us consider a discrete problem with the above modification of the convective term and with other terms being as in the conforming discretization. We shall consider spaces V_h^1, Q_h defined by (6), (8), respectively. In the triangular case, the space V_h^2 is defined as at the end of the previous section (i.e., it contains six bubbles per element). In the quadrilateral case, we use 32 bubbles per element [12] and we confine ourselves to rectangles. The right-hand side $\boldsymbol{f}$ is assumed to be piecewise linear or piecewise biquadratic. Then, in both cases, eliminating the space V_h^2 from the discretization, we recover the SUPG method by Franca and Frey [7], i.e.,

$$\nu a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + n(\boldsymbol{u}_h^1, \boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) - b(\boldsymbol{u}_h^1, q_h) + \\ + \sum_{T \in \mathcal{T}_h} \frac{\gamma_T h_T^2}{\nu} \int_T ((\nabla \boldsymbol{u}_h^1)\boldsymbol{u}_h^1 + \nabla p_h - \boldsymbol{f}) \cdot ((\nabla \boldsymbol{v}_h^1)\boldsymbol{u}_h^1 + \nabla q_h) \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}_h^1 \rangle.$$

This equivalence result is new for both the triangular and quadrilateral cases. In all other papers only linearized Navier–Stokes equations were considered due to the difficulties with the treatment of the convective term.

6 Conclusions

We have shown that a stabilization of the Stokes or Navier–Stokes equations often is equivalent to a modification of the conforming discretization together with the use of an appropriate supplementary space V_h^2. Such an equivalence is helpful for a better understanding of the properties of stabilized methods and for their theoretical investigations, e.g., in the framework of multigrid methods. Further, we have demonstrated that the technique of modified discretizations can help us to design new stabilized methods. The theory available for the modified discretizations then automatically provides existence and convergence statements for the corresponding stabilized methods.

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