A New Approach to Derive Stable Discretizations of the Stokes and Navier–Stokes Equations

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In this paper, we derive new stable finite element discretizations of the Stokes and Navier–Stokes equations. We start with an unstable pair of finite element spaces for approximating the velocity and pressure and enrich the velocity space by suitable functions, thus obtaining a stable pair of spaces. Then we drop some parts of the corresponding stiffness matrix and perform static condensation. The problem obtained in this way is considerably cheaper to solve than a conforming discretization with the enriched velocity space. The usual convergence statements valid for conforming discretizations with first order approximation spaces also hold for the stabilized problem and numerical experiments show that the errors of the discrete solutions are comparable to those ones obtained from a conforming discretization.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with a Lipschitz–continuous boundary $\partial\Omega$ and let us consider the incompressible Navier–Stokes equations

$$-\nu\,\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f}\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{u} = \boldsymbol{u}_b \;\text{ on } \partial\Omega\,, \quad (1)$$

where $\boldsymbol{u}$ is the velocity and p the pressure of a viscous fluid (with viscosity ν) contained in the domain Ω and exposed to an external body force $\boldsymbol{f}$. Denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \nu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x\,, \qquad n(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) = \int_\Omega \boldsymbol{v}\cdot(\nabla\boldsymbol{w})\boldsymbol{u}\,\mathrm{d}x\,,$$
$$b(\boldsymbol{v},p) = -\int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x\,,$$

we can introduce the usual weak formulation of (1): Given $\nu > 0$, $\boldsymbol{f} \in H^{-1}(\Omega)^d$ and $\boldsymbol{u}_b \in H^{\frac{1}{2}}(\partial\Omega)^d$, find $\boldsymbol{u} \in H^1(\Omega)^d$ and $p \in L^2_0(\Omega)$ such that

$$\boldsymbol{u} - \widetilde{\boldsymbol{u}}_b \in H^1_0(\Omega)^d\,, \quad (2)$$
$$a(\boldsymbol{u},\boldsymbol{v}) + n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \langle\boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in H^1_0(\Omega)^d\,, \quad (3)$$
$$b(\boldsymbol{u},q) = 0 \quad \forall\, q \in L^2_0(\Omega)\,, \quad (4)$$

where $\widetilde{\boldsymbol{u}}_b \in H^1(\Omega)^d$ is any extension of $\boldsymbol{u}_b$. If we drop the convective term

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$(\nabla \boldsymbol{u})\boldsymbol{u}$ in (1), we obtain the Stokes equations. The weak formulation of the Stokes equations is given by (2)–(4) without the term $n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})$.

Introducing some finite element spaces $\mathrm{V}_h \subset H_0^1(\Omega)^d$ and $\mathrm{Q}_h \subset L_0^2(\Omega)$, where h is a discretization parameter tending to zero, we can define a conforming finite element discretization of (2)–(4): Given an approximation $\widetilde{\boldsymbol{u}}_{bh} \in H^1(\Omega)^d$ of $\widetilde{\boldsymbol{u}}_b$, find $\boldsymbol{u}_h \in H^1(\Omega)^d$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathrm{V}_h\,, \tag{5}$$

$$a(\boldsymbol{u}_h,\boldsymbol{v}_h) + n(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p_h) = <\boldsymbol{f},\boldsymbol{v}_h> \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h\,, \tag{6}$$

$$b(\boldsymbol{u}_h,q_h) = 0 \quad \forall\, q_h \in \mathrm{Q}_h\,. \tag{7}$$

Existence and convergence results for the discretization (5)–(7) can be proven only if the spaces V_h and Q_h satisfy some variant of the Babuška–Brezzi condition

$$\exists\, \beta > 0\,: \quad \sup_{v_h \in \mathrm{V}_h \setminus \{0\}} \frac{b(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}} \geq \beta\, \|q_h\|_{0,\Omega} \quad \forall\, q_h \in \mathrm{Q}_h,\ h>0\,. \tag{8}$$

It is well known that, if this condition (or some its variant) is not satisfied, then the discrete problems are often not solvable or their solutions contain spurious oscillations. In such a case it is necessary to stabilize the discretization. One possibility is to add some extra stabilizing terms to the discretization obtained, e.g., using a Petrov–Galerkin method or a penalty method. Another possibility is to enlarge the velocity space so that the Babuška–Brezzi condition (8) holds. That, however, can significantly increase the computational cost. The idea of our approach is to start with a stable conforming discretization obtained by enriching a velocity space V_h^1 from an unstable pair of spaces V_h^1, Q_h by a suitable space V_h^2 and to derive a stabilized discretization by dropping certain terms from (5)–(7) with $\mathrm{V}_h = \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and by performing static condensation. In this general framework, stabilized discretizations for various unstable pairs of spaces can be obtained.

2 Stabilized Discretization

Since the spaces V_h^1, V_h^2 and Q_h are finite–dimensional, the problem (5)–(7) with $\mathrm{V}_h = \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ can be equivalently written as a system of algebraic equations for a finite set of unknown scalar parameters. We introduce some fixed bases $\{\boldsymbol{v}_{hi}^1\}_{i=1}^{N_h^1}$, $\{\boldsymbol{v}_{hi}^2\}_{i=1}^{N_h^2}$ and $\{q_{hi}\}_{i=1}^{N_h}$ in V_h^1, V_h^2 and Q_h, respectively, and, for any $\boldsymbol{u} \in H^1(\Omega)^d$ and $k,l \in \{1,2\}$, we define the matrices and vectors

$$\mathrm{A}_h^{kl} = \{a(\boldsymbol{v}_{hj}^l,\boldsymbol{v}_{hi}^k)\}_{i,j}\,, \qquad \mathrm{N}_h^{kl}(\boldsymbol{u}) = \{n(\boldsymbol{u},\boldsymbol{v}_{hj}^l,\boldsymbol{v}_{hi}^k)\}_{i,j}\,,$$

$$\mathrm{B}_h^k = \{b(\boldsymbol{v}_{hj}^k,q_{hi})\}_{i,j}\,, \qquad \mathrm{f}_h^{\mathrm{A}k} = \{<\boldsymbol{f},\boldsymbol{v}_{hi}^k> - a(\widetilde{\boldsymbol{u}}_{bh},\boldsymbol{v}_{hi}^k)\}_i\,,$$

$$\mathrm{f}_h^{\mathrm{N}k}(\boldsymbol{u}) = \{-n(\boldsymbol{u}, \widetilde{\boldsymbol{u}}_{bh}, \boldsymbol{v}_{hi}^k)\}_i\,, \qquad \mathrm{g}_h = \{-b(\widetilde{\boldsymbol{u}}_{bh}, q_{hi})\}_i\,.$$

Then the functions $\boldsymbol{u}_h$, p_h are a solution of the problem (5)–(7) if and only if

$$\begin{pmatrix} \mathrm{A}_h^{11} + \mathrm{N}_h^{11}(\boldsymbol{u}_h) & \mathrm{A}_h^{12} + \mathrm{N}_h^{12}(\boldsymbol{u}_h) & (\mathrm{B}_h^1)^T \\ \mathrm{A}_h^{21} + \mathrm{N}_h^{21}(\boldsymbol{u}_h) & \mathrm{A}_h^{22} + \mathrm{N}_h^{22}(\boldsymbol{u}_h) & (\mathrm{B}_h^2)^T \\ \mathrm{B}_h^1 & \mathrm{B}_h^2 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u}_h^1 \\ \mathrm{u}_h^2 \\ \mathrm{p}_h \end{pmatrix} = \begin{pmatrix} \mathrm{f}_h^{\mathrm{A}1} + \mathrm{f}_h^{\mathrm{N}1}(\boldsymbol{u}_h) \\ \mathrm{f}_h^{\mathrm{A}2} + \mathrm{f}_h^{\mathrm{N}2}(\boldsymbol{u}_h) \\ \mathrm{g}_h \end{pmatrix}, \quad (9)$$

where $\mathrm{u}_h^k = \{\mathrm{u}_{hi}^k\}_i$, $k = 1, 2$, and $\mathrm{p}_h = \{\mathrm{p}_{hi}\}_i$ are uniquely determined by

$$\sum_{j=1}^{N_h^1} \mathrm{u}_{hj}^1\, \boldsymbol{v}_{hj}^1 + \sum_{j=1}^{N_h^2} \mathrm{u}_{hj}^2\, \boldsymbol{v}_{hj}^2 = \boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh}\,, \qquad \sum_{j=1}^{N_h} \mathrm{p}_{hj}\, q_{hj} = p_h\,.$$

The matrix form of a discretization of the Stokes equations can be obtained by dropping $\mathrm{N}_h^{kl}(\boldsymbol{u}_h)$ and $\mathrm{f}_h^{\mathrm{N}k}(\boldsymbol{u}_h)$, $k, l = 1, 2$, in (9).

It can be shown (cf. [6]) that, for usual spaces V_h^2, the V_h^2 component of $\boldsymbol{u}_h$ can be dropped without influencing the convergence properties of $\boldsymbol{u}_h$ and, moreover, numerical experiments show that, dropping the V_h^2 component of $\boldsymbol{u}_h$, we often obtain a better approximation of $\boldsymbol{u}$. Having in mind that one of the basic reasons for introducing the space V_h^2 was to assure the solvability of the discrete problem, all this leads us to the idea to drop all the terms in (9) related to the space V_h^2 which are not important for the solvability of the discrete problem. Then we obtain the reduced system

$$\begin{pmatrix} \mathrm{A}_h^{11} + \mathrm{N}_h^{11}(\boldsymbol{u}_h^*) & 0 & (\mathrm{B}_h^1)^T \\ 0 & \mathrm{A}_h^{22} & (\mathrm{B}_h^2)^T \\ \mathrm{B}_h^1 & \mathrm{B}_h^2 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u}_h^1 \\ \mathrm{u}_h^2 \\ \mathrm{p}_h \end{pmatrix} = \begin{pmatrix} \mathrm{f}_h^{\mathrm{A}1} + \mathrm{f}_h^{\mathrm{N}1}(\boldsymbol{u}_h^*) \\ 0 \\ \mathrm{g}_h \end{pmatrix} \qquad (10)$$

with $\boldsymbol{u}_h^* = (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh})^1 + \widetilde{\boldsymbol{u}}_{bh}$, where $(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh})^1$ is the V_h^1 component of $\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh}$. Since the matrix A_h^{22} is regular, the vector u_h^2 can be eliminated and we arrive at the stabilized system

$$\begin{pmatrix} \mathrm{A}_h^{11} + \mathrm{N}_h^{11}(\boldsymbol{u}_h^*) & (\mathrm{B}_h^1)^T \\ \mathrm{B}_h^1 & \mathrm{S}_h \end{pmatrix} \begin{pmatrix} \mathrm{u}_h^1 \\ \mathrm{p}_h \end{pmatrix} = \begin{pmatrix} \mathrm{f}_h^{\mathrm{A}1} + \mathrm{f}_h^{\mathrm{N}1}(\boldsymbol{u}_h^*) \\ \mathrm{g}_h \end{pmatrix}, \qquad (11)$$

where $\mathrm{S}_h = -\mathrm{B}_h^2\, (\mathrm{A}_h^{22})^{-1}\, (\mathrm{B}_h^2)^T$. In many cases, the matrix A_h^{22} is diagonal and hence the matrix S_h can be easily evaluated. The vector u_h^2 need not to be computed since it does not influence the asymptotic behaviour of the discrete solution.

The first stabilized discretization of the type (11) was introduced in [3] for the Stokes equations and extended to the Navier–Stokes equations in [7]. In these papers, the matrix S_h corresponds to a discretization of $(\nabla p, \nabla q)$

and is therefore positive definite. However, in our approach, the matrix S_h is indefinite in general.

3 Theoretical Results

We assume that we are given a family of spaces V_h^1, $\mathrm{V}_h^2 \subset H_0^1(\Omega)^d$, $\mathrm{Q}_h \subset L_0^2(\Omega)$, where h is a positive parameter tending to zero. We assume that $\mathrm{V}_h^1 \cap \mathrm{V}_h^2 = \{\mathbf{0}\}$ and that there exist operators $r_h \in \mathcal{L}(H^2(\Omega)^d \cap H_0^1(\Omega)^d, \mathrm{V}_h^1)$ and $s_h \in \mathcal{L}(L_0^2(\Omega), \mathrm{Q}_h)$ satisfying

$$\|\boldsymbol{v} - r_h \boldsymbol{v}\|_{1,\Omega} \leq C\,h\,\|\boldsymbol{v}\|_{2,\Omega} \qquad \forall\ \boldsymbol{v} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d\,, \tag{12}$$

$$\|q - s_h q\|_{0,\Omega} \leq C\,h\,\|q\|_{1,\Omega} \qquad \forall\ q \in H^1(\Omega) \cap L_0^2(\Omega)\,. \tag{13}$$

Further we suppose that the spaces $\mathrm{V}_h \equiv \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and Q_h satisfy the Babuška–Brezzi condition (8). For any $\boldsymbol{v}_h \in \mathrm{V}_h$ we denote by $\boldsymbol{v}_h^1 \in \mathrm{V}_h^1$ and $\boldsymbol{v}_h^2 \in \mathrm{V}_h^2$ the uniquely determined functions satisfying $\boldsymbol{v}_h^1 + \boldsymbol{v}_h^2 = \boldsymbol{v}_h$ and we assume that

$$\|\boldsymbol{v}_h^1\|_{1,\Omega} + \|\boldsymbol{v}_h^2\|_{1,\Omega} \leq C\,\|\boldsymbol{v}_h\|_{1,\Omega} \qquad \forall\ \boldsymbol{v}_h \in \mathrm{V}_h\,, \tag{14}$$

$$\|\boldsymbol{v}_h^2\|_{0,\Omega} \leq C\,h\,|\boldsymbol{v}_h^2|_{1,\Omega} \qquad \forall\ \boldsymbol{v}_h^2 \in \mathrm{V}_h^2\,. \tag{15}$$

Finally, we assume that $\|\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega} \to 0$ for $h \to 0$. The convergence results in the next theorem will be stated for the reduced system (10) with a more general right–hand side generated by a functional $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$.

Theorem 3.1 *For any $\boldsymbol{f}_h \in H^{-1}(\Omega)^d$, the reduced system (10) without $\mathrm{N}_h^{11}(\boldsymbol{u}_h^*)$ and $\mathrm{f}_h^{\mathrm{N}1}(\boldsymbol{u}_h^*)$ (i.e., the reduced system corresponding to the Stokes equations) has a unique solution $\boldsymbol{u}_h$, p_h and, if $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{[\mathrm{V}_h]'} \to 0$, then we have $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \to 0$ for $h \to 0$, where $\boldsymbol{u}$, p is the weak solution of the Stokes equations. Further, if $\boldsymbol{u}$, $\widetilde{\boldsymbol{u}}_b \in H^2(\Omega)^d$, $p \in H^1(\Omega)$, $\|\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega} \leq C\,h$ and $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{[\mathrm{V}_h]'} \leq C\,h$, then $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C\,h$. If, in addition, Ω is such that the Stokes problem is regular and if $\|\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{0,\Omega} + \|\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{0,\partial\Omega} \leq C\,h^2$ and $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{[\mathrm{V}_h^1]'} \leq C\,h^2$, then we obtain $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq C\,h^2$.*

All the convergence results are also valid with $\boldsymbol{u}_h$ replaced by $\boldsymbol{u}_h^*$ (cf. below (10)) provided that (14) is satisfied for $\|\cdot\|_{0,\Omega}$ and (12) holds for $\|\cdot\|_{0,\Omega}$ with h^2.

Proof. See [6]. □

Remark 3.1 Setting

$$<\boldsymbol{f}_h, \boldsymbol{v}_h> = (\boldsymbol{f}, \boldsymbol{v}_h^1) \qquad \forall\ \boldsymbol{v}_h \in \mathrm{V}_h \tag{16}$$

for $\boldsymbol{f} \in L^2(\Omega)^d$, we obtain a functional satisfying all the assumptions from Theorem 3.1 and yielding the right–hand side of (10). However, also other definitions of $\boldsymbol{f}_h$ are possible.

For investigating the properties of the reduced system (10) for the Navier–Stokes equations, we assume, in addition, that there exist an operator $z_h \in \mathcal{L}(H_0^1(\Omega)^d, H^1(\Omega)^d)$ and a constant $\alpha < 1$ independent of h such that

$$z_h \boldsymbol{v}_h = \boldsymbol{v}_h^1 \quad \forall\, \boldsymbol{v}_h \in \mathrm{V}_h\,, \qquad \|z_h \boldsymbol{v}\|_{1,\Omega} \le C\, \|\boldsymbol{v}\|_{1,\Omega} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d\,,$$

$$\|\boldsymbol{v} - z_h \boldsymbol{v}\|_{0,\Omega} + h^\alpha\, \|\boldsymbol{v} - z_h \boldsymbol{v}\|_{0,4,\Omega} \le C\, h\, \|\boldsymbol{v}\|_{1,\Omega} \qquad \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d\,,$$

$$\|\boldsymbol{v} - z_h \boldsymbol{v}\|_{0,\Omega} + h\, |\boldsymbol{v} - z_h \boldsymbol{v}|_{1,\Omega} \le C\, h^2\, \|\boldsymbol{v}\|_{2,\Omega} \qquad \forall\, \boldsymbol{v} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d\,.$$

Further, we assume that $\|\boldsymbol{f} - \boldsymbol{f}_h\|_{-1,\Omega} \to 0$ for $h \to 0$. Then, applying results of the theory of approximation of branches of nonsingular solutions (cf. [5], Chapter IV, Section 3), we can prove that, for any nonsingular weak solution of the Navier–Stokes equations (1), there exists a neighbourhood where the discrete problems corresponding to the reduced system (10) are uniquely solvable for small values of h and where the discrete solutions converge to the weak solution. Under the assumptions of Theorem 3.1, we can prove the same convergence rates as for the Stokes equations. Details can be found in [6]. For both the Stokes and the Navier–Stokes equations, the above optimal convergence rates have been confirmed by many numerical experiments.

4 Examples of Finite Element Spaces

We assume that we are given a regular simplicial or quadrilateral, resp. hexahedral triangulation $\mathcal{T}_h$ of Ω such that the union of its elements is equal to $\overline{\Omega}$. Depending on the type of triangulation, the space V_h^1 consists of continuous piecewise linear or bilinear, resp. trilinear functions. The simplest space Q_h is the space of P_0 functions (piecewise constant functions) from $L_0^2(\Omega)$ and it is easy to show that the spaces V_h^1 and Q_h generally do not satisfy the Babuška–Brezzi condition (8). For $\mathcal{T}_h$ consisting of triangles, the space V_h^2 assuring the fulfilment of (8) can be defined as linear hull of piecewise quadratic vector functions in directions of normal vectors to edges having their supports in triangles adjacent to the edges (cf. [2]). Analogously we can proceed for quadrilaterals (cf. [4]). For these choices of V_h^2, which we shall denote by $\mathrm{V}_h^{2,\,\mathrm{nond}}$, the matrix A_h^{22} is not diagonal. However, by simple modifications (cf. [6]), spaces $\mathrm{V}_h^{2,\,\mathrm{diag}}$ leading to diagonal matrices can be obtained. For $\mathcal{T}_h$ made up of triangles, we can also use Q_h consisting of P_1 functions (continuous piecewise linear functions) from $L_0^2(\Omega)$ and V_h^2 generated by cubic bubble fuctions, which is the so–called mini element (cf. [1]). All the five definitions of V_h^2 have their three–dimensional counterparts which can be obtained in a straightforward way, and all ten pairs of the spaces V_h, Q_h obtained in such a way satisfy all the assumptions from Section 3.

Table 1: Numbers of nonzero elements of A_h and B_h.

Q_h	$\mathcal{T}_h$	$\#\mathrm{B}_h$	$\#\mathrm{A}_h$ for conform		$\#\mathrm{A}_h$ for reduced	
			$\mathrm{V}_h^{2,\,\mathrm{nond}}$	$\mathrm{V}_h^{2,\,\mathrm{diag}}$	$\mathrm{V}_h^{2,\,\mathrm{nond}}$	$\mathrm{V}_h^{2,\,\mathrm{diag}}$
P_0	triangles	18	77 (70)	65 (58)	29 (22)	17 (10)
P_0	squares	12	80 (71)	68 (59)	32 (23)	20 (11)
P_0	tetrahedra	96	489 (459)	417 (387)	129 (99)	57 (27)
P_0	cubes	30	330 (276)	300 (246)	114 (60)	84 (30)
P_1	triangles	26	42 (21)		18 (9)	
P_1	tetrahedra	117	207 (69)		63 (21)	

Table 1 shows the asymptotic numbers of nonzero elements of the stiffness matrices B_h and A_h for the above pairs of V_h and Q_h in case of uniform triangulations of a cube. Due to the special structure of the matrix A_h, only a part of its nonzero elements has to be stored in a computer memory. The numbers of the elements which are actually stored are given in parentheses. All the values in Tab. 1 have to be multiplied by the number of nodes of the respective triangulation. The matrix A_h consists of $\mathrm{A}_h^{kl} + \mathrm{N}_h^{kl}(\boldsymbol{u}_h)$, $k, l = 1, 2$, in case of conforming discretizations and of $\mathrm{A}_h^{11} + \mathrm{N}_h^{11}(\boldsymbol{u}_h^*)$ and A_h^{22} in case of reduced discretizations. The table shows that the saving of the computer memory and the computational operations due to the use of the reduced discrete problems is substantial. In case of Q_h made up of P_0 functions, a further saving is achieved using the spaces $\mathrm{V}_h^{2,\,\mathrm{diag}}$ instead of $\mathrm{V}_h^{2,\,\mathrm{nond}}$. For example, for tetrahedral triangulations (cf. the third row of Tab. 1), we save about 94 % of the computer memory needed for saving the matrix A_h and about 78 % of the memory needed for saving the whole matrix (consisting of A_h and B_h). Let us mention that, storing the matrix S_h instead of the matrices B_h^2 and A_h^{22}, we save further memory in case of the mini element so that the total memory saving is 49 % (for triangles) and 66 % (for tetrahedra).

5 Numerical Results for the Navier–Stokes Equations

In this section we present a few numerical results for the Navier–Stokes equations with $\nu = 1$ defined in $\Omega = (0, 1)^3$. The triangulation of Ω consists of tetrahedra and is obtained by dividing Ω into 30^3 equal cubes and by dividing

Table 2: Errors of the discrete solutions of the Navier–Stokes equations with $\nu = 1$ and the solution (17).

	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	$\|\boldsymbol{u}-\boldsymbol{u}_h^*\|_{0,\Omega}$	$\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{1,\Omega}$	$\vert\boldsymbol{u}-\boldsymbol{u}_h^*\vert_{1,\Omega}$	$\|p-p_h\|_{0,\Omega}$
D1	1.11 e−2	1.16 e−2	2.11 e+0	2.11 e+0	1.33 e+0
D2	1.44 e−2	1.55 e−2	2.96 e+0	2.13 e+0	2.53 e+0
D3	1.20 e−2	1.22 e−2	2.35 e+0	2.12 e+0	2.95 e+0
D4	1.15 e−2	1.18 e−2	2.15 e+0	2.11 e+0	2.20 e+0
D5	1.18 e−2	1.19 e−2	2.31 e+0	2.11 e+0	2.94 e+0

each cube into six tetrahedra. We shall consider the following five types of discretizations of the Navier–Stokes equations:

- D1 conforming discretization with $\mathrm{V}_h^2 = \mathrm{V}_h^{2,\,\mathrm{nond}}$;
- D2 reduced discretization with $\mathrm{V}_h^2 = \mathrm{V}_h^{2,\,\mathrm{nond}}$ and $\boldsymbol{f}_h = \boldsymbol{f}$;
- D3 reduced discretization with $\mathrm{V}_h^2 = \mathrm{V}_h^{2,\,\mathrm{diag}}$ and $\boldsymbol{f}_h = \boldsymbol{f}$;
- D4 reduced discretization with $\mathrm{V}_h^2 = \mathrm{V}_h^{2,\,\mathrm{nond}}$ and $\boldsymbol{f}_h$ satisfying (16);
- D5 reduced discretization with $\mathrm{V}_h^2 = \mathrm{V}_h^{2,\,\mathrm{diag}}$ and $\boldsymbol{f}_h$ satisfying (16).

The space V_h^1 consists of P_1 functions and Q_h of P_0 functions.

First, we consider the Navier–Stokes equations with the solution

$$\boldsymbol{u}(x) = 30\,(x_1\,x_2^2 + x_3^3,\, x_1^2\,x_3,\, x_1^3 - x_2^2\,x_3)\,, \qquad p(x) = 2\,x_1^2 + x_2\,x_3\,. \tag{17}$$

The errors of the discrete solutions are presented in Tab. 2, where again $\boldsymbol{u}_h^* = (\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh})^1 + \widetilde{\boldsymbol{u}}_{bh}$. In all five cases, the velocity errors are similar and the pressure errors increase for the reduced discretizations. In case of the solution

$$\boldsymbol{u}(x) = 30\,(x_2,\, x_3,\, x_1)\,, \qquad p(x) = x_1\,, \tag{18}$$

large errors arise if the right–hand side in the second equation of (10) is not dropped (cf. Tab. 3). That is caused by the large convective term, which is present in the function $\boldsymbol{f}$ but has no counterpart on the left–hand side of the second equation. Therefore, the function $\boldsymbol{f}$ should not be used as the right–hand side of the second equation in reduced discretizations of the Navier–Stokes

Table 3: Errors of the discrete solutions of the Navier–Stokes equations with $\nu = 1$ and the solution (18).

	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$	$\|\boldsymbol{u} - \boldsymbol{u}_h^*\|_{0,\Omega}$	$\lvert\boldsymbol{u} - \boldsymbol{u}_h\rvert_{1,\Omega}$	$\lvert\boldsymbol{u} - \boldsymbol{u}_h^*\rvert_{1,\Omega}$	$\|p - p_h\|_{0,\Omega}$
D1	1.07 e−5	1.31 e−5	2.36 e−3	1.49 e−4	7.13 e−3
D2	8.78 e−3	9.58 e−3	1.98 e+0	1.21 e−1	5.45 e−1
D3	1.87 e−3	1.94 e−3	4.13 e−1	2.45 e−2	2.22 e−1
D4	2.03 e−5	2.51 e−5	4.61 e−3	2.78 e−4	7.20 e−3
D5	1.79 e−5	1.88 e−5	3.87 e−3	2.14 e−4	6.92 e−3

equations. In case of the Stokes equations, the errors are similar for all five discretizations also for the linear solution. From Tab. 3, we further clearly see that, in all five cases, we obtain a significantly better discrete solution if we drop the V_h^2 component of $\boldsymbol{u}_h$. The numbers of iterations needed for computing the discrete solutions with a given precision were nearly the same for all five discretizations and hence the computational time was smallest for the discretizations D3 and D5. Thus, among the discretizations D1–D5, the discretization D5 seems to be the best choice. As a final conclusion, we can say that the reduced discretizations produce discrete solutions which have a similar accuracy as in the conforming case, but which can be obtained with a much smaller effort.

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