New nonconforming finite elements for solving the incompressible Navier–Stokes equations

P. Knobloch

Summary. We introduce a new class of nonconforming finite elements suitable for an accurate approximation of convection-dominated effects and of divergence-free functions and hence also well-suited for the numerical solution of the incompressible Navier–Stokes equations.

1 Introduction

The aim of this paper is to introduce a new class of nonconforming finite elements suitable for the numerical solution of the incompressible Navier–Stokes equations

$$\alpha \frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\nabla \boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{3}$$

These equations describe the velocity $\boldsymbol{u}$ and the pressure p in an incompressible viscous fluid contained in a domain Ω. We assume that Ω is a bounded domain in $\mathbb{R}^2$ and that its boundary $\partial\Omega$ is polygonal. The parameter $\nu > 0$ denotes the kinematic viscosity, $\boldsymbol{f}$ is an outer volume force and $\alpha \in \{0, 1\}$ determines whether the equations are stationary or non-stationary. In the latter case we have to add an initial condition.

Probably the most popular and best understood numerical method for solving the equations (1)–(3) is the finite element method (cf., e.g., [4, 9]). It is well-known that, in view of the incompressibility constraint (2), finite element spaces $\mathbf{V}_h$ and Q_h for approximating the velocity $\boldsymbol{u}$ and pressure p, respectively, cannot be chosen arbitrarily if one wants to obtain a stable discretization with respect to $h \to 0$ and no additional stabilization of the equation (2) is used (see, e.g., [1, 4] for details). A sufficient requirement on the spaces $\mathbf{V}_h$, Q_h is the validity of the inf–sup condition

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h \setminus \{\boldsymbol{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta \, \|q_h\|_{0,\Omega} \qquad \forall\, q_h \in \mathrm{Q}_h, \tag{4}$$

where $\beta > 0$ is independent of the discretization parameter h,

$$b_h(\boldsymbol{v}_h, q_h) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x, \qquad |\boldsymbol{v}_h|_{1,h} = \Bigg(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \Bigg)^{1/2}$$

and $\mathcal{T}_h$ is a triangulation of Ω consisting of elements K used for constructing the spaces $\mathbf{V}_h$ and Q_h. The inf–sup condition (4) makes it possible to establish optimal error estimates for the discrete solution of (1)–(3).

In this paper we consider finite element spaces $\mathbf{V}_h$ of nonconforming type, which means that they contain functions which are continuous across the edges of the elements only in some weak sense. One of the advantages of such spaces in comparison with their conforming counterparts is that the inf–sup condition (4) usually holds with more convenient pressure spaces Q_h. Moreover, discretely divergence-free bases can often easily be constructed in the nonconforming case, which is not true for conforming finite element spaces. A further advantage of nonconforming elements is that they are more suitable for a parallel implementation than conforming elements since they enable a cheap local communication between processors.

However, it was already observed in [5] that nonconforming finite elements can lead to very inaccurate numerical results if they are applied to the numerical solution of convection-dominated problems. The reason is that, in certain cases, the nonconformity of the finite element spaces may lead to additional consistency errors or to the loss of uniform coercivity of the bilinear form in the discrete problem. This phenomenon may also occur when solving the Navier–Stokes equations (1)–(3) since, in typical applications, the viscosity ν is small and hence the convective term in (1) dominates the diffusion term.

In order to see more clearly what happens in the convection-dominated regime, we consider the scalar convection-diffusion equation

$$-\nu \Delta u + \boldsymbol{b} \cdot \nabla u + c\, u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5}$$

which is a standard model problem for investigating convection-dominated effects. Note that (5) can be viewed as a simplification of the time-discretized equations (1)–(3). In this case $\operatorname{div} \boldsymbol{b} = 0$ and $c = \alpha/\tau$, where τ is a positive constant. Theoretical results for a nonconforming finite element discretization of (5) indicate possible failures and provide a motivation for introducing new nonconforming finite elements for which the same error estimates can be derived as for their conforming counterparts. Since these elements will be obtained by modifying the standard nonconforming elements, we call them P_n^{mod} elements. These elements lead to very accurate and robust discretizations of the convection-diffusion equation (5) and satisfy the inf–sup condition (4) for various pressure spaces. That makes the P_n^{mod} elements very well-suited for the numerical solution of the incompressible Navier–Stokes equations (1)–(3).

The plan of the paper is as follows. First, in Sect. 2, we introduce notation needed in the subsequent sections. Then, in Sect. 3, we present theoretical results for a nonconforming streamline diffusion finite element discretization of (5). These results motivate the definition of the P_n^{mod} element in Sect. 4. Finally, in Sect. 5, we briefly discuss for which pressure spaces the spaces $\mathbf{V}_h$ defined using the P_n^{mod} element satisfy the inf–sup condition (4).

2 Notation

We assume that we are given a family $\{\mathcal{T}_h\}$ of triangulations of the domain Ω parametrized by a positive parameter $h \to 0$. Each triangulation $\mathcal{T}_h$ consists of a finite number of shape-regular closed triangular elements K satisfying $h_K \equiv \operatorname{diam}(K) \leq$

h and possessing the usual compatibility properties (see [2]). We denote by $\mathcal{E}_h$ the set of edges E of $\mathcal{T}_h$, by $\mathcal{E}_h^i$ the set of inner edges, and by h_E the length of the edge E. For any edge E, we choose a fixed unit normal vector $\boldsymbol{n}_E$ to E. Consider any $E \in \mathcal{E}_h^i$ and let K, $\widetilde{K}$ be the two elements possessing the edge E denoted in such a way that $\boldsymbol{n}_E$ points into $\widetilde{K}$. If v is a function belonging to the space

$$H^{1,h}(\Omega) = \{v \in L^2(\Omega);\ v|_K \in H^1(K)\ \ \forall\ K \in \mathcal{T}_h\},$$

then we define the jump of v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E.$$

If $E \subset \partial\Omega$, then we set $[|v|]_E = v|_E$.

Throughout the paper we use standard notation for the usual function spaces and norms (see, e.g., [2]). The scalar product in the space $L^2(G)$ will be denoted by $(\cdot\,,\cdot)_G$ and we set $(\cdot\,,\cdot) = (\cdot\,,\cdot)_\Omega$. Finally, we denote by C a generic constant independent of h and ν.

3 Application of nonconforming finite elements to a scalar convection-diffusion equation

In this section, we establish a nonconforming finite element discretization of the convection-diffusion equation (5) and discuss its properties. We assume that $\nu \in (0, 1)$ is constant and that $\boldsymbol{b} \in W^{1,\infty}(\Omega)^2$, $c \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. In addition, we make the standard assumption

$$c - \frac{1}{2}\operatorname{div}\boldsymbol{b} \geq c_0 \geq 0, \tag{6}$$

where c_0 is constant. We assume that

$$c = \operatorname{div}\boldsymbol{b} = 0 \qquad \text{if } c_0 = 0. \tag{7}$$

As we know from Sect. 1, the assumptions (6) and (7) are satisfied if (5) corresponds to a simplification of the Navier–Stokes equations (1)–(3).

A natural weak formulation of the convection-diffusion equation (5) reads:

find $u \in H_0^1(\Omega)$ such that

$$\nu\,(\nabla u, \nabla v) + (\boldsymbol{b}\cdot\nabla u, v) + (c\,u, v) = (f, v) \qquad \forall\ v \in H_0^1(\Omega). \tag{8}$$

Due to (6), this weak formulation has a unique solution. We shall approximate the space $H_0^1(\Omega)$ by a nonconforming finite element space V_h such that

$$P_n(K) \cap (H_0^1(\Omega)|_K) \subset \mathrm{V}_h|_K \subset H^2(K) \qquad \forall\ K \in \mathcal{T}_h, \tag{9}$$

where $n \geq 1$ is a given integer. An example of a nonconforming space V_h satisfying (9) is the standard nonconforming space of order n (cf. [3])

$$V_h^{nc,n} = \Big\{ v \in L^2(\Omega);\ v|_K \in P_n(K)\ \ \forall\, K \in \mathcal{T}_h,$$
$$\int_E [|v|]_E\, q\, d\sigma = 0\ \ \forall\, q \in P_{n-1}(E),\ E \in \mathcal{E}_h \Big\}.$$

The usual way to obtain a finite element discretization of (8) is to define the terms in (8) involving derivatives elementwise and to replace the space $H_0^1(\Omega)$ by the space V_h. However, to obtain a well-posed problem, we first integrate by parts in the term $(\boldsymbol{b} \cdot \nabla u, v)$, which gives the bilinear form

$$a_h^s(u, v) = \frac{1}{2} \sum_{K \in \mathcal{T}_h} [(\boldsymbol{b} \cdot \nabla u, v)_K - (\boldsymbol{b} \cdot \nabla v, u)_K - (\operatorname{div} \boldsymbol{b}, u\, v)_K].$$

In addition, to stabilize the discrete problem in convection-dominated regions, we introduce a streamline diffusion term

$$a_h^{sd}(u, v) = \sum_{K \in \mathcal{T}_h} (-\nu\, \Delta u + \boldsymbol{b} \cdot \nabla u + c\, u,\ \delta_K\, \boldsymbol{b} \cdot \nabla v)_K,$$

where δ_K is a control parameter. We assume that

$$0 \leq \delta_K, \qquad 2\, \|c\|_{0,\infty,K}^2\, \delta_K \leq c_0, \qquad 2\, \nu\, \mu_1^2\, \delta_K \leq h_K^2, \tag{10}$$

where μ_1 is any constant for which the inverse inequality

$$\|\Delta\, v_h\|_{0,K} \leq \mu_1\, h_K^{-1}\, |v_h|_{1,K} \qquad \forall\, v_h \in V_h,\ K \in \mathcal{T}_h$$

holds. Thus, we obtain the discrete problem:

find $u_h \in V_h$ such that

$$\nu \sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_K + a_h^s(u_h, v_h) + (c\, u_h, v_h) + a_h^{sd}(u_h, v_h)$$
$$= (f, v_h) + \sum_{K \in \mathcal{T}_h} (f,\ \delta_K\, \boldsymbol{b} \cdot \nabla v_h)_K \qquad \forall\, v_h \in V_h. \tag{11}$$

Under the assumptions (10), the bilinear form represented by the left-hand side of (11) is coercive (cf. [5]) with respect to the streamline diffusion norm

$$|||v||| = \Big(\sum_{K \in \mathcal{T}_h} \{\nu\, |v|_{1,K}^2 + c_0\, \|v\|_{0,K}^2 + \delta_K\, \|\boldsymbol{b} \cdot \nabla v\|_{0,K}^2\} \Big)^{1/2}$$

and hence the discrete problem (11) has a unique solution.

Theorem 1. *Suppose that the assumptions* (9) *and* (10) *are satisfied and that the space* V_h *satisfies the patch test of order* $k+1$, *i.e.,*

$$\int_E [|v_h|]_E \, q \, d\sigma = 0 \qquad \forall\, v_h \in V_h,\ q \in P_k(E),\ E \in \mathcal{E}_h, \tag{12}$$

where $k \geq n-1$ *is a given integer. Let the weak solution of* (5) *belong to* $H^{\max\{n+1,k+1\}}(\Omega)$ *and let* $\boldsymbol{b} \in W^{k+1,\infty}(\Omega)^2$. *Set* $m = \max\{2, k+1\}$. *Then the discrete solution* u_h *satisfies*

$$|||u-u_h||| \leq C\,h^n \left(\sum_{K\in\mathcal{T}_h} \gamma_K \, |u|^2_{n+1,K} \right)^{1/2} + C\,h^k \left(\sum_{E\in\mathcal{E}_h} \gamma_E \, \|u\|^2_{m,S_E} \right)^{1/2}, \tag{13}$$

where S_E *is the union of the elements adjacent to* E *and*

$$\gamma_K = \nu + h_K^2 + \delta_K + (\max\{\nu, \delta_K\})^{-1} h_K^2, \qquad \gamma_E = \max\left\{ \frac{\nu}{h_E^2}, c_0 \right\}^{-1}.$$

Proof. The theorem can be proved by using the techniques of [5] and [8].

The above estimate suggests setting

$$\delta_K = \begin{cases} \kappa_K\, h_K & \text{if } h_K > \nu, \\ 0 & \text{if } h_K \leq \nu, \end{cases}$$

where κ_K assures the validity of (10) and satisfies $0 < \kappa_0 \leq \kappa_K \leq \kappa_1$ with κ_0, κ_1 independent of h. Then $\gamma_K \leq C\,(\nu + h_K)$ and hence in the convection-dominated case ($\nu \leq h$) we have $\gamma_K \leq C\,h$. Since the sum over edges in (13) stems only from the nonconformity, we obtain for $V_h \subset H_0^1(\Omega)$ the well-known optimal ν-uniform estimate

$$|||u-u_h||| \leq C\,h^{n+1/2}\,|u|_{n+1,\Omega}.$$

For a general nonconforming space V_h satisfying the assumptions of Theorem 1, a ν-uniform estimate can be obtained only if $c_0 > 0$. Then we get

$$|||u-u_h||| \leq C\,h^{n+1/2}\,|u|_{n+1,\Omega} + C\,h^k \|u\|_{\max\{2,k+1\},\Omega}. \tag{14}$$

Thus, in using the standard nonconforming space $V_h^{\mathrm{nc},n}$, the ν-uniform convergence order is only $n-1$. In particular, for the Crouzeix–Raviart element ($V_h = V_h^{\mathrm{nc},1}$), the ν-uniform convergence order is 0, which was also confirmed by numerical experiments (cf. [8]). On the other hand, the estimate (14) shows that, for obtaining the optimal convergence order $n+1/2$, it is sufficient to require that V_h satisfies (12) with $k = n+1$, i.e., the patch test of order $n+2$. We shall construct such spaces in the next section.

If $c_0 = 0$, a ν-uniform estimate cannot be derived in general. However, it is clear from (13) that a higher patch test will increase the accuracy of the discrete solution.

4 Definition and properties of the P_n^{mod} element

The aim of this section is to construct nonconforming finite element spaces satisfying a patch test of a higher order than in the standard case. Note that it is not possible to require this higher patch test from the standard nonconforming spaces $\mathrm{V}_h^{\mathrm{nc},n}$ since then these spaces would become conforming. Therefore, we will modify the spaces $\mathrm{V}_h^{\mathrm{nc},n}$ which will lead to a new finite element called the P_n^{mod} element.

To introduce the P_n^{mod} element, we first describe the respective functions on the standard reference element $\widehat{K}$. The idea is to enrich the space $P_n(\widehat{K})$ corresponding to the space $\mathrm{V}_h^{\mathrm{nc},n}$ by suitable functions such that we can obtain a nonconforming finite element space satisfying the patch test of a given order. Our basic requirement is that this procedure must not destroy the edge-oriented structure of the space $\mathrm{V}_h^{\mathrm{nc},n}$. It turns out that, independently of n and of the required order of the patch test, it suffices to add three functions $\widehat{b}_1$, $\widehat{b}_2$ and $\widehat{b}_3$ associated respectively with the edges $\widehat{E}_1$, $\widehat{E}_2$ and $\widehat{E}_3$ of the element $\widehat{K}$. This gives the space

$$P_n^{\mathrm{mod}}(\widehat{K}) = P_n(\widehat{K}) \oplus \mathrm{span}\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\}.$$

We assume that

$$\widehat{b}_1 \in H^1(\widehat{K}), \qquad \widehat{b}_1|_{\partial\widehat{K}\setminus\widehat{E}_1} = 0, \tag{15}$$

$$\widehat{b}_1|_{\widehat{E}_1} \text{ is odd with respect to the midpoint of } \widehat{E}_1, \tag{16}$$

$$\int_{\widehat{E}_1} [(1 - 2\widehat{\lambda}_2) + \widehat{b}_1]\,\widehat{q}\,\mathrm{d}\widehat{\sigma} = 0 \qquad \forall\, \widehat{q} \in P_k(\widehat{E}_1), \tag{17}$$

where $k \geq n$ and $\widehat{\lambda}_2$ is the barycentric coordinate on $\widehat{K}$ with respect to the vertex $\widehat{x}_2$ which is the vertex of $\widehat{K}$ opposite the edge $\widehat{E}_2$ (the remaining vertices $\widehat{x}_1$ and $\widehat{x}_3$ and the barycentric coordinates $\widehat{\lambda}_1$ and $\widehat{\lambda}_3$ are defined analogously). The functions $\widehat{b}_2$ and $\widehat{b}_3$ are simply defined by affine transformations of $\widehat{b}_1$, i.e., $\widehat{b}_i = \widehat{b}_1 \circ \widehat{F}_i$, $i = 2, 3$, where $\widehat{F}_2$ and $\widehat{F}_3$ are affine regular mappings on $\mathbb{R}^2$ such that $\widehat{F}_i(\widehat{K}) = \widehat{K}$, $\widehat{F}_i(\widehat{E}_i) = \widehat{E}_1$, $i = 2, 3$.

Note that the above assumptions are not sufficient for using the P_n^{mod} element in a streamline diffusion discretization since the assumptions do not allow us to apply second derivatives to the function $\widehat{b}_1$. Thus, we further assume that $\widehat{b}_1$ belongs to H^2, at least with respect to some subdivision of $\widehat{K}$.

Examples of the function $\widehat{b}_1$ satisfying (15)–(17) for various values of k include the following functions:

$$k = 2: \qquad \widehat{b}_1 = 10\widehat{\lambda}_2\widehat{\lambda}_3(\widehat{\lambda}_2 - \widehat{\lambda}_3), \tag{18}$$

$$k = 4: \qquad \widehat{b}_1 = (28\widehat{\lambda}_2\widehat{\lambda}_3 - 126\widehat{\lambda}_2^2\widehat{\lambda}_3^2)(\widehat{\lambda}_2 - \widehat{\lambda}_3), \tag{19}$$

$$k = 6: \qquad \widehat{b}_1 = (54\widehat{\lambda}_2\widehat{\lambda}_3 - 594\widehat{\lambda}_2^2\widehat{\lambda}_3^2 + 1716\widehat{\lambda}_2^3\widehat{\lambda}_3^3)(\widehat{\lambda}_2 - \widehat{\lambda}_3). \tag{20}$$

Generally, to satisfy the assumptions (15)–(17) with $k = 2l$, where l is a given integer, we can set

$$\widehat{b}_1 = (\widehat{\lambda}_2 - \widehat{\lambda}_3) \sum_{i=1}^{l} a_i \, (\widehat{\lambda}_2 \widehat{\lambda}_3)^i, \tag{21}$$

where $a_1, \ldots, a_l \in \mathbb{R}$ are uniquely determined numbers.

For introducing a finite element space we have to specify a set of so-called nodal functionals which have to be unisolvent with the space $P_n^{\mathrm{mod}}(\widehat{K})$. For example, we can use $n + 1$ moments on each edge, i.e.,

$$\frac{1}{|\widehat{E}_i|} \int_{\widehat{E}_i} \widehat{v} \widehat{\lambda}_{i+1}^j \, \mathrm{d}\widehat{\sigma}, \qquad j = 0, \ldots, n, \qquad i = 1, 2, 3,$$

where $\widehat{\lambda}_4 \equiv \widehat{\lambda}_1$. If $n > 2$, then we further add a set of nodal functionals unisolvent with the space $P_n(\widehat{K}) \cap H_0^1(\widehat{K})$. These nodal functionals may be, e.g., suitable integrals over $\widehat{K}$ or values in the inner points of the principal lattice of order n of the triangle $\widehat{K}$.

For any element $K \in \mathcal{T}_h$, we introduce a regular affine mapping $F_K : \widehat{K} \to K$ such that $F_K(\widehat{K}) = K$ and we transform the space $P_n^{\mathrm{mod}}(\widehat{K})$ onto the space

$$P_n^{\mathrm{mod}}(K) = \left\{ \widehat{v} \circ F_K^{-1};\ \widehat{v} \in P_n^{\mathrm{mod}}(\widehat{K}) \right\}.$$

Similarly, we transform the nodal functionals defined on the space $P_n^{\mathrm{mod}}(\widehat{K})$ onto analogous nodal functionals defined on the spaces $P_n^{\mathrm{mod}}(K)$. Now, the P_n^{mod} finite element space consists of functions which belong to $P_n^{\mathrm{mod}}(K)$ on each element K and for which the nodal functionals assigned to boundary edges vanish and the corresponding pairs of nodal functionals assigned to inner edges give the same value. Thus, we obtain the space

$$\mathrm{V}_h^{\mathrm{mod},n} = \Big\{ v \in L^2(\Omega);\ v|_K \in P_n^{\mathrm{mod}}(K) \ \ \forall\, K \in \mathcal{T}_h, \\ \int_E [|v|]_E \, q \, \mathrm{d}\sigma = 0 \ \ \forall\, q \in P_n(E),\ E \in \mathcal{E}_h \Big\}.$$

As we see, the space $\mathrm{V}_h^{\mathrm{mod},n}$ is of approximation order n and satisfies the patch test of order $n + 1$. It is a nonconforming finite element space which is edge- and element-oriented, i.e., one can use basis functions which are all associated with edges or with elements. Precisely, for any inner edge, we have $n+1$ basis functions associated with this edge and having their supports in the two elements adjacent to this edge. Further, on each element, we have $\frac{1}{2}(n-1)(n-2)$ basis functions having their supports in this element.

To investigate further properties of the space $V_h^{\mathrm{mod},n}$, we first introduce some notation. For any $K \in \mathcal{T}_h$ and any $E \in \mathcal{E}_h$ such that $E \subset \partial K$, we set

$$b_{K,E} = \begin{cases} \widehat{b}_i \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

where $i \in \{1, 2, 3\}$ is chosen in such a way that $E = F_K(\widehat{E}_i)$. Then

$$P_n^{\mathrm{mod}}(K) = P_n(K) \oplus \operatorname{span}\{b_{K,E}|_K\}_{E \in \mathcal{E}_h,\, E \subset \partial K}.$$

Further, for any edge $E \in \mathcal{E}_h^i$, we denote by ζ_E the standard basis function of $V_h^{\mathrm{nc},1}$ associated with the edge E (i.e., ζ_E is piecewise linear, equals 1 on E and vanishes at the midpoints of all edges different from E). Finally, we denote by K, $\widetilde{K}$ the two elements adjacent to E, by E, E_1, E_2 the edges of K, and by E, E_3, E_4 the edges of $\widetilde{K}$. Now, we define functions ψ_E, $\chi_E \in V_h^{\mathrm{mod},n}$ by

$$\psi_E = \zeta_E + \beta_{E,1}\, b_{K,E_1} + \beta_{E,2}\, b_{K,E_2} + \beta_{E,3}\, b_{\widetilde{K},E_3} + \beta_{E,4}\, b_{\widetilde{K},E_4}, \tag{22}$$

$$\chi_E = b_{K,E} + \beta_{E,5}\, b_{\widetilde{K},E}, \tag{23}$$

where the coefficients $\beta_{E,1}, \ldots, \beta_{E,5}$ are uniquely determined and equal to 1 or -1. In view of the assumptions on $\widehat{b}_1$, we have $\chi_E \in H_0^1(\Omega)$ and hence the functions χ_E generate a conforming subspace of $V_h^{\mathrm{mod},n}$. The functions ψ_E are purely nonconforming functions since they have jumps across the edges $E_1, \ldots, E_4$ and they can be viewed as modified basis functions of $V_h^{\mathrm{nc},1}$. It is obvious that the functions $\{\psi_E, \chi_E\}_{E \in \mathcal{E}_h^i}$ are linearly independent and hence they represent a basis of the space $V_h^{\mathrm{mod},1}$. If $n > 1$, then

$$V_h^{\mathrm{mod},n} = \operatorname{span}\{\psi_E, \chi_E\}_{E \in \mathcal{E}_h^i} \oplus \{v \in H_0^1(\Omega);\ v|_K \in P_n(K)\ \ \forall\, K \in \mathcal{T}_h,\ v(x) = 0 \text{ at any vertex } x \text{ of } \mathcal{T}_h\}.$$

This immediately follows from the fact that the two spaces on the right-hand side are linearly independent and that the sum of their dimensions is equal to $\dim V_h^{\mathrm{mod},n}$. Thus, we see that the space $V_h^{\mathrm{mod},n}$ always contains a basis in which the only nonconforming functions are the functions ψ_E and the supplementary functions generated by the function $\widehat{b}_1$ appear only in the functions ψ_E and χ_E. Therefore, to investigate the order of the patch test satisfied by the space $V_h^{\mathrm{mod},n}$, it suffices to deal with the functions ψ_E.

Consider any edge $E \in \mathcal{E}_h^i$ and let E_i, $i \in \{1, 2\}$, be one of the edges used in (22). Further, we denote by F_{E_i} any linear mapping which maps $\widehat{E}_1$ onto E_i. Then, applying (17), we deduce for any $q \in P_k(E_i)$ that

$$\begin{aligned} \int_{E_i} [|\psi_E|]_{E_i}\, q \,\mathrm{d}\sigma &= \int_{E_i} [|\zeta_E + \beta_{E,i}\, b_{K,E_i}|]_{E_i}\, q \,\mathrm{d}\sigma \\ &= \pm \frac{|E_i|}{|\widehat{E}_1|} \int_{\widehat{E}_1} [(1 - 2\widehat{\lambda}_2) + \widehat{b}_1]\, q \circ F_{E_i} \,\mathrm{d}\widehat{\sigma} = 0. \end{aligned}$$

The same holds for $i \in \{3, 4\}$ with b_{K,E_i} replaced by $b_{\widetilde{K},E_i}$. Consequently,

$$\int_E [|v_h|]_E \, q \, \mathrm{d}\sigma = 0 \qquad \forall \, v_h \in \mathrm{V}_h^{\mathrm{mod},n}, \; q \in P_k(E), \; E \in \mathcal{E}_h,$$

i.e., the space $\mathrm{V}_h^{\mathrm{mod},n}$ satisfies the patch test of order $k+1$. Thus, we can conclude that, choosing the function $\widehat{b}_1$ in a suitable way, we can enforce an arbitrarily high order of the patch test satisfied by the space $\mathrm{V}_h^{\mathrm{mod},n}$. Therefore, we can always recover the optimal convergence rates known from the conforming finite element method when we use the space $\mathrm{V}_h^{\mathrm{mod},n}$ in (11). Note also that there is no difference between defining the P_n^{mod} elements of odd and even approximation orders whereas standard triangular nonconforming elements of even degree are difficult to handle. A further difference from standard nonconforming elements is that the spaces $\mathrm{V}_h^{\mathrm{mod},n}$ with various values of n can be made nested by defining them by using the same function $\widehat{b}_1$. Finally, we mention that any P_n^{mod} element satisfies the discrete Korn inequality [6], which is not true for most nonconforming elements of first order.

We applied the P_n^{mod} elements to the numerical solution of various convection-diffusion problems and the results were always better than for the corresponding standard nonconforming spaces. Particularly, in the convection-dominated regime, the accuracy of P_n^{mod} solutions was often higher by several orders of magnitude. The P_n^{mod} element always led to optimal ν-uniform convergence rates with respect to all usual norms whereas standard nonconforming spaces very often gave only suboptimal convergence rates and sometimes they did not lead to any ν-uniform convergence. We refer to [8] for results of numerical experiments comparing the P_1^{mod} element and the Crouzeix–Raviart element.

5 Validity of the inf–sup condition

In this section we briefly discuss for which pressure spaces Q_h the inf–sup condition (4) holds if $\mathbf{V}_h = [\mathrm{V}_h^{\mathrm{mod},n}]^2$. Since continuous piecewise polynomial functions of degree n are contained in the space $\mathrm{V}_h^{\mathrm{mod},n}$, we can use a space Q_h consisting of continuous piecewise polynomial functions of degree $n-1$ (provided that $n > 1$). If $n > 3$ we can even use discontinuous piecewise polynomial functions of degree $n-1$. This follows from the theory of Taylor–Hood elements and the results of Scott and Vogelius which hold under additional assumptions on the triangulations (cf. [1] and [4] and their references).

For the P_1^{mod} element it was recently shown (cf. [7]) that the inf–sup condition holds with Q_h consisting of discontinuous piecewise linear functions. The proof is based on the macroelement technique and uses the assumptions that any element $K \in \mathcal{T}_h$ has at least one vertex in Ω and that $\widehat{b}_1 \in L_0^2(\widehat{K})$, which is satisfied for the examples of $\widehat{b}_1$ given in (18)–(21). Using the macroelement technique and the assumptions mentioned, one can also prove that the inf–sup condition holds for the P_n^{mod} element with $n \in \{2, 3\}$ if Q_h consists of continuous piecewise polynomial functions of degree n.

We note that an optimal convergence rate with respect to $h \to 0$ is guaranteed when we use the P_n^{mod} element together with a pressure space Q_h of piecewise polynomial degree $n-1$. However, it may often pay to use Q_h of a higher degree than $n-1$ since a better pressure space can help to increase the accuracy of the discrete solution if the viscosity ν is small.

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References

[1] Brezzi, F., Fortin, M. (1991): Mixed and hybrid finite element methods. Springer, New York

[2] Ciarlet, P.G. (1991): Basic error estimates for elliptic problems. In: Ciarlet, P.G., Lions, J.-L. (eds.): Handbook of numerical analysis. vol. II. Finite element methods. Part I. North-Holland, Amsterdam, pp. 17–351

[3] Crouzeix, M., Raviart, P.-A. (1973): Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. Rev. Française Automat. Informat. Recherche Opérationelle Sér. Rouge **7** (R-3), 33–76

[4] Girault, V., Raviart, P.-A. (1986): Finite element methods for Navier–Stokes equations. Theory and algorithms. Springer, Berlin

[5] John, V., Maubach, J.M., Tobiska, L. (1997): Nonconforming streamline-diffusion-finite-element-methods for convection-diffusion problems. Numer. Math. **78**, 165–188

[6] Knobloch, P. (2000): On Korn's inequality for nonconforming finite elements. Tech. Mech. **20**, 205–214; Errata, ibid. 375

[7] Knobloch, P. (2001): On the inf–sup condition for the P_1^{mod} element. Preprint MATH-KNM-2001/4. Charles University, Prague

[8] Knobloch, P., Tobiska, L. (1999): The P_1^{mod} element: a new nonconforming finite element for convection-diffusion problems. Preprint 99-28. Fakultät für Mathematik, Otto-von-Guericke-Universität, Magdeburg

[9] Temam, R. (1977): Navier–Stokes equations. Theory and numerical analysis. North-Holland, Amsterdam