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Discrete Friedrichs' and Korn's Inequalities in Two and Three Dimensions

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Abstract — The uniform validity of various forms of the discrete Friedrichs' and Korn's inequalities used in the finite element method is proven for polygonal (resp. polyhedral) approximations of the domain and for continuous functions generated by a finite–dimensional subspace of $W^{1,\infty}(\widehat{T})$. For the Friedrichs' inequalities, the uniform validity in $H^1(\Omega_h)$ is proven as well.

Keywords: finite element method, discrete Friedrichs' inequality, discrete Korn's inequality.

AMS(MOS) subject classification: 65N30.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a bounded domain with a Lipschitz–continuous boundary $\partial\Omega$ and let $\Gamma \subset \partial\Omega$ have a positive $(N-1)$–dimensional measure. Consider the space

$$W = \{\, u \in H^1(\Omega)\,,\ u = 0 \text{ on } \Gamma \,\}\,.$$

Then the following *Friedrichs' inequality* holds (cf. [5])

$$\|u\|_{1,\Omega} \le K_1(\Omega, \Gamma)\, |u|_{1,\Omega} \qquad \forall\, u \in W\,,$$

where the constant $K_1(\Omega, \Gamma)$ depends only on Ω and Γ.

Let Ω_h be a polygonal or polyhedral domain approximating Ω and let $\Gamma_h \subset \partial\Omega_h$ approximate Γ. Let $W_h \subset C(\Omega_h)$ be a finite–dimensional space approximating W which is supposed to be constructed by the finite element method and to satisfy

$$W_h \subset \{\, v \in H^1(\Omega_h)\,,\ v = 0 \text{ on } \Gamma_h \,\}\,.$$

Then we obtain the following *discrete Friedrichs' inequality*

$$\|v\|_{1,\Omega_h} \le K_1(\Omega_h, \Gamma_h)\, |v|_{1,\Omega_h} \qquad \forall\, v \in W_h\,, \tag{1.1}$$

where the constant $K_1(\Omega_h, \Gamma_h)$ again depends only on Ω_h and Γ_h.

In this paper we prove that the constants $K_1(\Omega_h, \Gamma_h)$ are bounded for h tending to zero, which is very important for proving convergence statements in the theory of the finite element method (cf. e.g. [4]). This problem was already investigated in a more general fashion (for Ω_h with a curved boundary and for both the inequality (1.1) and the corresponding Friedrichs' inequality in $H^2(\Omega_h)$) in [7], however, only for the

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two–dimensional case. The idea in [7] was to use the following type of the Friedrichs' inequality (cf. [5])

$$\|u\|_{1,\Omega}^2 \le \widetilde{K}_1(\Omega,\Gamma)\left\{|u|_{1,\Omega}^2 + \int_\Gamma u^2\,\mathrm{d}\sigma\right\} \qquad \forall\, u \in H^1(\Omega) \tag{1.2}$$

for a suitable *extension* of $v \in W_h$ onto $\Omega \cup \Omega_h$ (clearly, if $\Omega \subset \Omega_h$, no extension has to be constructed). Another but similar approach based on the so–called Zlámal's ideal interpolation was used in [3]. In the three–dimensional case, however, it is not possible to use such techniques. Therefore, in this paper, we develop a new method consisting in constructing a *modification* $\widetilde{v}$ of the function $v \in W_h$ by stretching the domain Ω_h together with the function v (cf. Fig. 1 and relations (2.7) and (2.8) in the next section). The crucial point is that the stretched domain $\widetilde{\Omega}_h \supset \Omega_h$ differ from Ω_h superlinearly with respect to h. This makes it possible to prove the boundedness of $K_1(\Omega_h,\Gamma_h)$ for both the two–dimensional and the three–dimensional cases. In the same manner, the boundedness of the constants $K_2(\Omega_h,\Gamma_h)$ in the *discrete Korn's inequalities*

$$\|\boldsymbol{v}\|_{1,\Omega_h}^2 \le K_2(\Omega_h,\Gamma_h)\int_{\Omega_h}(\nabla\,\boldsymbol{v}+\nabla\,\boldsymbol{v}^{\mathrm{T}})\cdot(\nabla\,\boldsymbol{v}+\nabla\,\boldsymbol{v}^{\mathrm{T}})\,\mathrm{d}x \qquad \forall\,\boldsymbol{v}\in[W_h]^N \tag{1.3}$$

can be shown. Under slightly stronger assumptions than needed for investigating (1.1) and (1.3), we shall also prove the boundedness of the constants in the discrete analogue of (1.2) and in the corresponding form of the discrete Korn's inequality. Finally, we shall show that the results proven for the discrete Friedrichs' inequalities also remain valid for functions from $H^1(\Omega_h)$.

It should be mentioned that the use of the presented method is not restricted to proofs of discrete Friedrichs' and Korn's inequalities and the paper provides a rather general and simple tool for proving the uniform validity of discrete analogues of various relations valid in $H^1(\Omega)$. As a further example, a discrete trace theorem is proven using this method in the proof of Corollary 4.2.

The basic techniques we use are similar to those used in [7] for investigating the two–dimensional case. However, less restrictive assumptions on the triangulation are needed for our proof, particularly, there are no global limitations on the ratio of diameters of any two elements of the triangulation. Hence, the results are also valid for nonuniformly refined meshes (obtained e.g. by adaptive refinement).

The notation used in this paper is standard. We only mention that $\|\cdot\|_{k,p,\Omega}$ (resp. $|\cdot|_{k,p,\Omega}$) is the norm (resp. seminorm) in the Sobolev space $W^{k,p}(\Omega)$, $k = 0,1,\ldots$, $p \in \langle 1,\infty\rangle$. For $p = 2$, the second index is dropped (i.e., we write $\|\cdot\|_{k,\Omega}$, resp. $|\cdot|_{k,\Omega}$) and the Sobolev space is denoted by $H^k(\Omega)$.

2. ASSUMPTIONS AND NOTATION

As already mentioned in the preceding section,

(A1) $\Omega \subset \mathbb{R}^N$, $N = 2,3$, is a bounded domain with a Lipschitz–continuous boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ has a positive $(N-1)$–dimensional measure.

Now we can define a *triangulation* of the domain Ω (cf. also [2]). Given $h > 0$, we say that the set $\mathcal{T}_h$ is a triangulation of the domain Ω if $\mathcal{T}_h$ consists of closed triangles (if $N = 2$) or tetrahedra (if $N = 3$) called elements and denoted by T, $\mathrm{diam}(T) \le h$ for any $T \in \mathcal{T}_h$, all vertices of any $T \in \mathcal{T}_h$ belong to $\overline{\Omega}$, vertices lying on $\partial\Omega_h$ belong to $\partial\Omega$, and any two different elements T_1, $T_2 \in \mathcal{T}_h$ are either disjoint or possess either a common vertex or a common edge or (if $N = 3$) a common face. (We used the notation $\mathrm{diam}(T) \equiv \sup_{x,y\in T} |x-y|$ for the diameter of T.)

We denote

$$\begin{aligned} \Omega_h &= \operatorname{int} \bigcup_{T\in\mathcal{T}_h} T\,, \\ h_T &= \mathrm{diam}(T)\,, \\ \varrho_T &= \sup_{B\subset T \text{ is a ball}} \mathrm{diam}(B)\,, \end{aligned}$$

where $\operatorname{int} A$ is the interior of the set A. The length of any edge of an element T is clearly from the interval $(\varrho_T, h_T>$.

(A2) All triangulations of Ω used in this paper form a *regular family* $\boldsymbol{\mathcal{T}} = \{\mathcal{T}_h\}$, i.e., there exists a number $\sigma > 0$ such that for any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ and any $T \in \mathcal{T}_h$

$$\frac{h_T}{\varrho_T} \le \sigma\,. \tag{2.1}$$

Any element T used hereafter is supposed to satisfy the relation (2.1).

The assumptions (A1) and (A2) are standard. For our special purpose, however, we also make the following rather complex assumption (cf. also Fig. 1), the validity of which is investigated in Section 6.

(A3) There exists a constant $C > 1$ with the following properties. Denote

$$h_0 = \frac{1}{C^2\,(16\,\sigma)^{N-1}}\,. \tag{2.2}$$

Then, for any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ with $h \in (0, h_0)$, the vertices lying on $\partial\Omega$ can be moved outside Ω in such a way that vertices of any $T \in \mathcal{T}_h$ move at most Ch_T^2 (the vertices lying in Ω remain fixed), any $T \in \mathcal{T}_h$ transforms onto $\widetilde{T}$ (for T with vertices in Ω, we have $\widetilde{T} = T$), the set

$$\widetilde{\mathcal{T}}_h = \{\widetilde{T}\}_{T\in\mathcal{T}_h}$$

is a triangulation of the domain

$$\widetilde{\Omega}_h = \operatorname{int} \bigcup_{\widetilde{T}\in\widetilde{\mathcal{T}}_h} \widetilde{T}\,,$$

and

$$\Omega \subset \widetilde{\Omega}_h\,, \quad \Omega_h \subset \widetilde{\Omega}_h\,. \tag{2.3}$$

In addition, for any $\widetilde{T} \in \widetilde{\mathcal{T}}_h$ and for any $x \in \widetilde{T} \cap \partial\Omega$, we suppose that

$$\mathrm{dist}(x, \widetilde{T} \cap \partial\widetilde{\Omega}_h) < Ch_T^2\,, \tag{2.4}$$

where $\mathrm{dist}(x, A) \equiv \inf_{y\in A} |x-y|$ is the distance between x and the set A.

Remark 2.1. The assumption (A3) ensures that the relative differences between T and $\widetilde{T}$ and between $\widetilde{T}$ and $\widetilde{T} \cap \Omega$ linearly converge to zero when $h \to 0$. Nevertheless, the *linear* convergence is not important for our futher considerations and it would also be sufficient to replace Ch_T^2 by $Ch_T^{1+\alpha}$ (and h_0 by h_0^α) with some fixed $\alpha > 0$. Then the auxiliary results in the next section remain valid when the exponents of h and h_T in (3.2), (3.15)–(3.17), (3.19)–(3.21), (3.23) and (3.26) are decreased by $1-\alpha$, which implies that all the results of Section 4 and Section 5 remain valid as well. The choice $\alpha = 1$ is considered only for simplicity.

It is suitable to define the following sets (cf. Fig. 1)

$$
\begin{aligned}
\mathcal{I}_h &= \{T \in \mathcal{T}_h\,,\ T \cap \partial\Omega_h = \emptyset\,\}\,,\\
\mathcal{B}_h &= \{T \in \mathcal{T}_h\,,\ T \cap \partial\Omega_h \neq \emptyset\,\}\,,\\
\widetilde{\mathcal{B}}_h &= \{\widetilde{T} \in \widetilde{\mathcal{T}}_h\,,\ \widetilde{T} \cap \partial\widetilde{\Omega}_h \neq \emptyset\,\}\,,\\
\Omega_h^I &= \operatorname{int} \bigcup_{T \in \mathcal{I}_h} T\,, \quad \Omega_h^B = \operatorname{int} \bigcup_{T \in \mathcal{B}_h} T\,, \quad \widetilde{\Omega}_h^B = \operatorname{int} \bigcup_{\widetilde{T} \in \widetilde{\mathcal{B}}_h} \widetilde{T}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{T}_h &= \mathcal{I}_h \cup \mathcal{B}_h \quad \text{and} \quad \mathcal{I}_h \cap \mathcal{B}_h = \emptyset\,,\\
\widetilde{\mathcal{T}}_h &= \mathcal{I}_h \cup \widetilde{\mathcal{B}}_h \quad \text{and} \quad \mathcal{I}_h \cap \widetilde{\mathcal{B}}_h = \emptyset\,,\\
\Omega_h &= \operatorname{int} \overline{\Omega_h^I \cup \Omega_h^B} \quad \text{and} \quad \Omega_h^I \cap \Omega_h^B = \emptyset\,, && (2.5)\\
\widetilde{\Omega}_h &= \operatorname{int} \overline{\Omega_h^I \cup \widetilde{\Omega}_h^B} \quad \text{and} \quad \Omega_h^I \cap \widetilde{\Omega}_h^B = \emptyset\,. && (2.6)
\end{aligned}
$$

The last assumption is related to the approximation of Γ. We suppose that

(A4) the set $\Gamma_h \subset \partial\Omega_h$ consists of whole edges (if $N = 2$) or faces (if $N = 3$) and, for any $T \in \mathcal{T}_h$, we have $\widetilde{T} \cap \Gamma \neq \emptyset \Rightarrow T \cap \Gamma_h \neq \emptyset$, i.e., at least one vertex of T belongs to Γ_h if $\widetilde{T} \cap \Gamma \neq \emptyset$.

Remark 2.2. The assumption (A4) corresponds to the common conception of a good approximation of a part of the boundary. (A slightly stronger formulation is the following one: if some edge or face of $\partial\Omega_h$ does not belong to Γ_h, then its neighbourhood of order $O(h_T^2)$ does not contain any points of Γ.) In addition, if $\partial\Gamma$ is piecewise C^2 and $\mathcal{T}_h$ is a usual triangulation where any vertex belonging to $\partial\Gamma_h$ lies on $\partial\Gamma$ (and vice versa), then (A4) always holds for sufficiently small h (that can be readily shown using the Taylor formula).

Now, let $\widehat{T}$ be the standard reference triangle or tetrahedron and $F_T : \mathbb{R}^N \to \mathbb{R}^N$ the regular affine mapping which maps $\widehat{T}$ onto $T \in \mathcal{T}_h$ (cf. e.g. [2]). For any $v \in C(\overline{\Omega}_h)$ and $T \in \mathcal{T}_h$, we introduce the notation

$$
\hat{v}_T = v \circ (F_T|_{\widehat{T}})\,. \tag{2.7}
$$

Let $\widehat{W} \subset W^{1,\infty}(\widehat{T})$ be some fixed *finite–dimensional* space. Then we define the spaces

$$
V_h = \{\, v \in C(\overline{\Omega}_h)\,,\ \hat{v}_T \in \widehat{W} \quad \forall\, T \in \mathcal{T}_h\,\}\,, \qquad W_h = \{\, v \in V_h\,,\ v = 0 \text{ on } \Gamma_h\,\}\,.
$$

Since $\widehat{W}$ is finite–dimensional, there exists a set $\widehat{A} \subset \widehat{T}$ with $\operatorname{dist}(\widehat{A}, \partial\widehat{T}) > 0$ such that

$$
\dim \widehat{W}|_{\widehat{A}} = \dim \widehat{W} \qquad \text{and} \qquad 1 \in \widehat{W}|_{\widehat{A}} \Leftrightarrow 1 \in \widehat{W}\,,
$$

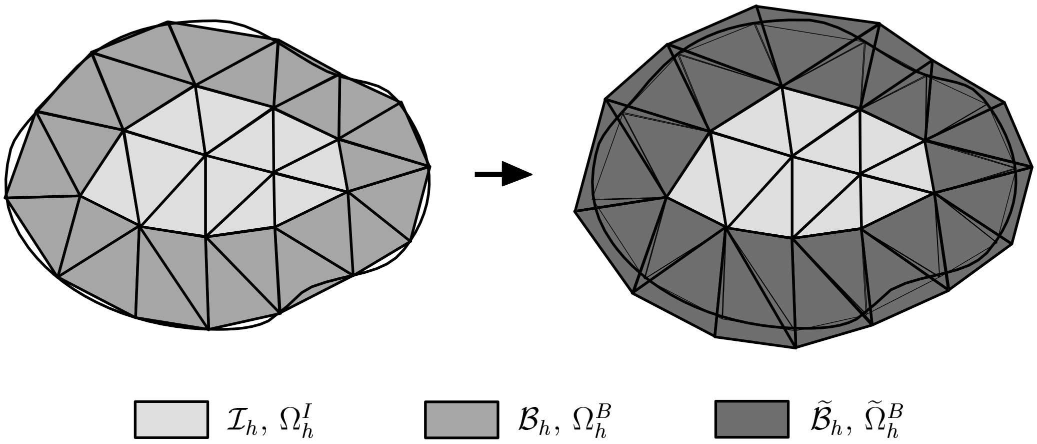

Figure 1. Construction of a modification of the domain Ω_h.

where the space $\widehat{W}|_{\widehat{A}}$ consists of functions from $\widehat{W}$ restricted onto $\widehat{A}$. Further, if the assumption (A3) is fulfilled, then it also remains valid when the constant C is arbitrarily increased. Therefore, without loss of generality, we can assume that

$$\mathrm{dist}(\widehat{A}, \partial\widehat{T}) > \frac{1}{12\,C}\,.$$

Finally, if $v \in V_h$ and $\widetilde{\mathcal{T}}_h$ is a modified triangulation corresponding to $\mathcal{T}_h$, then we define the function $\widetilde{v} \in C(\overline{\widetilde{\Omega}}_h)$ by

$$\widetilde{v}|_{\widetilde{T}} = \widehat{v}_T \circ (F_{\widetilde{T}}^{-1}|_{\widetilde{T}}) \quad \forall\, \widetilde{T} \in \widetilde{\mathcal{T}}_h\,. \tag{2.8}$$

Clearly,

$$\widetilde{v} = v \ \text{ in } \ \Omega_h^I\,. \tag{2.9}$$

The function $\widetilde{v}$ is the modification of v mentioned in Section 1. The assumption (A3) ensures that $\widetilde{v}$ and v differ only "very slightly" for h sufficiently small (cf. the lemmas in the next section).

3. AUXILIARY RESULTS

Lemma 3.1. *(i) There exists $\widehat{T}_0 \subset \widehat{T}$ with $\widehat{A} \subset \widehat{T}_0$ and $\mathrm{meas}_N(\widehat{T}_0) > 0$ such that*

$$F_{\widetilde{T}}(\widehat{T}_0) \subset \Omega \quad \forall\, \widetilde{T} \in \widetilde{\mathcal{B}}_h\,,\ h \in (0, h_0)\,. \tag{3.1}$$

(ii) For any $h \in (0, h_0)$, there exists $\widehat{\Sigma}_h \subset \widehat{T}$ such that

$$\mathrm{meas}_N(\widehat{\Sigma}_h) \le 4\,C\,\sigma\,h \tag{3.2}$$

and

$$\widetilde{T} \setminus F_{\widetilde{T}}(\widehat{\Sigma}_h) \subset \Omega \quad \forall\, \widetilde{T} \in \widetilde{\mathcal{B}}_h\,. \tag{3.3}$$

Proof. Set $q = 4\,C\sigma h/3 < 4\,C\sigma h_0/3 \le 1/(12\,C)$ and let $\widehat{T}_q^* \subset \widehat{T}$ be a triangle with vertices (q, q), $(1 - 2\,q, q)$, $(q, 1 - 2\,q)$, resp. a tetrahedron with vertices (q, q, q), $(1 - 3\,q, q, q)$, $(q, 1 - 3\,q, q)$, $(q, q, 1 - 3\,q)$. Then, for any $\widetilde{T} \in \widetilde{\mathcal{B}}_h$, $\mathrm{dist}(F_{\widetilde{T}}(\widehat{T}_q^*), \partial\widetilde{T}) > q\varrho_{\widetilde{T}} > Ch_T^2$ and hence, according to (2.4), $F_{\widetilde{T}}(\widehat{T}_q^*) \subset \Omega$. Thus, we obtain the lemma setting $\widehat{T}_0 = \widehat{T}_{1/(12\,C)}^*$ and $\widehat{\Sigma}_h = \widehat{T} \setminus \widehat{T}_q^*$. $\square$

Lemma 3.2. *There exist constants c_1, c_2 depending only on $\widehat{T}_0$ and $\widehat{W}$ such that for any measurable set $\widehat{G} \subset \widehat{T}$ and any function $\widehat{v} \in \widehat{W}$*

$$\|\widehat{v}\|_{0,\widehat{G}}^2 \le c_1 \,\mathrm{meas}_N(\widehat{G})\, \|\widehat{v}\|_{0,\widehat{T}_0}^2 , \tag{3.4}$$

$$|\widehat{v}|_{1,\widehat{G}}^2 \le c_2 \,\mathrm{meas}_N(\widehat{G})\, |\widehat{v}|_{1,\widehat{T}_0}^2 . \tag{3.5}$$

Proof. Let $d = \dim \widehat{W}$ and let the functions $\{\widehat{b}_i\}_{i=1}^d \subset \widehat{W}$ form a basis in $\widehat{W}$. If $1 \in \widehat{W}$, we set $\widehat{b}_1 = 1$. Consider any $\widehat{v} \in \widehat{W}$. Then there exist uniquely determined real numbers $\{\alpha_i\}_{i=1}^d$ such that

$$\widehat{v} = \sum_{i=1}^d \alpha_i \,\widehat{b}_i .$$

Denoting

$$D = d \max_{i=1,\dots,d} \|\widehat{b}_i\|_{1,\infty,\widehat{T}}^2 ,$$

we obtain for any measurable set $\widehat{G} \subset \widehat{T}$

$$\|\widehat{v}\|_{0,\widehat{G}}^2 \le D \,\mathrm{meas}_N(\widehat{G}) \sum_{i=1}^d {\alpha_i}^2 , \qquad |\widehat{v}|_{1,\widehat{G}}^2 \le D \,\mathrm{meas}_N(\widehat{G}) \sum_{i=1}^d {\alpha_i}^2 . \tag{3.6}$$

Since $\widehat{A} \subset \widehat{T}_0$, the norm $\|\cdot\|_{0,\widehat{T}_0}$ is a norm on $\widehat{W}$ and it follows from the equivalence of norms on finite–dimensional spaces that

$$\widetilde{c} \sum_{i=1}^d {\alpha_i}^2 \le \|\widehat{v}\|_{0,\widehat{T}_0}^2 , \tag{3.7}$$

where the constant $\widetilde{c}$ depends only on $\widehat{T}_0$ and $\widehat{b}_1,\dots,\widehat{b}_d$. Thus, (3.4) follows from (3.6) and (3.7).

If $1 \notin \widehat{W}$, then the seminorm $|\cdot|_{1,\widehat{T}_0}$ is a norm on $\widehat{W}$ and (3.6) together with the equivalence of norms on $\widehat{W}$ implies (3.5). If $1 \in \widehat{W}$, then $\widehat{b}_1 = 1$ and we have

$$|\widehat{v}|_{1,\widehat{G}}^2 \le D \,\mathrm{meas}_N(\widehat{G}) \sum_{i=2}^d {\alpha_i}^2 . \tag{3.8}$$

According to Theorem 14.1 in [2], p. 120, there exists a constant $\overline{c} = \overline{c}(\widehat{T}_0)$ such that

$$|u|_{1,\widehat{T}_0}^2 \ge \overline{c} \inf_{\beta \in \mathbf{R}} \|u + \beta\|_{0,\widehat{T}_0}^2 \qquad \forall\, u \in H^1(\widehat{T}_0) .$$

Using (3.7), we obtain

$$\inf_{\beta \in \mathbf{R}} \|\widehat{v} + \beta\|_{0,\widehat{T}_0}^2 = \inf_{\beta \in \mathbf{R}} \Big\|(\alpha_1 + \beta)\,\widehat{b}_1 + \sum_{i=2}^d \alpha_i \,\widehat{b}_i\Big\|_{0,\widehat{T}_0}^2 \ge \widetilde{c} \inf_{\beta \in \mathbf{R}} \Big[(\alpha_1 + \beta)^2 + \sum_{i=2}^d {\alpha_i}^2\Big]$$

and hence

$$|\widehat{v}|_{1,\widehat{T}_0}^2 \ge \overline{c}\,\widetilde{c} \sum_{i=2}^d {\alpha_i}^2 ,$$

which together with (3.8) implies (3.5). □

Lemma 3.3. *Let $T \in \mathcal{T}_h$ with $h \in (0, h_0)$ be any element belonging to the family of regular triangulations and let $F_{\widetilde{T}} : \mathbb{R}^N \to \mathbb{R}^N$ be the regular affine mapping corresponding to the modified element $\widetilde{T}$. Let $\widehat{G} \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz–continuous boundary and let $\widehat{v} \in H^1(\widehat{G})$. Define $G = F_{\widetilde{T}}(\widehat{G})$ and $\widetilde{v} = \widehat{v} \circ (F_{\widetilde{T}}^{-1}|_G)$. Then*

$$c_3\, h_T^N\, \|\widehat{v}\|^2_{0,\widehat{G}} \leq \|\widetilde{v}\|^2_{0,G} \leq 2\, h_T^N\, \|\widehat{v}\|^2_{0,\widehat{G}}\,, \tag{3.9}$$

$$c_4\, h_T^{N-2}\, |\widehat{v}|^2_{1,\widehat{G}} \leq |\widetilde{v}|^2_{1,G} \leq c_5\, h_T^{N-2}\, |\widehat{v}|^2_{1,\widehat{G}}\,, \tag{3.10}$$

where the constants c_3, c_4 and c_5 depend only on N and σ.

Proof. Denoting

$$J_T = \left|\det \frac{DF_T}{D\,\widehat{x}}\right|\,, \qquad J_{\widetilde{T}} = \left|\det \frac{DF_{\widetilde{T}}}{D\,\widehat{x}}\right|\,,$$

we obtain

$$\frac{h_T^N}{\sigma^{N-1}} \leq J_T \leq h_T^N\,, \qquad \left|\left(\frac{DF_T{}^{-1}}{D\,x}\right)_{i,j}\right| \leq \frac{(N-1)!\, h_T^{N-1}}{J_T}\,, \quad i,j = 1,\ldots,N\,. \tag{3.11}$$

According to (A3), the distance between the corresponding vertices of T and $\widetilde{T}$ is at most $h_T/16$ and therefore

$$\frac{7}{8}\, h_T \leq h_{\widetilde{T}} \leq \frac{9}{8}\, h_T\,. \tag{3.12}$$

Since the elements of $DF_T/D\,\widehat{x}$ and $DF_{\widetilde{T}}/D\,\widehat{x}$ differ at most $2\,Ch_T^2$, we obtain using the definition of the determinant

$$|J_T - J_{\widetilde{T}}| \leq N!\, N\, 2\, Ch_T^2\, (h_T + 2\, Ch_T^2)^{N-1} \leq \overline{c}(N,C)\, h_T^{N+1}\,. \tag{3.13}$$

Thus, by (2.2) and (3.11), $|J_T - J_{\widetilde{T}}| \leq 2\, J_T/3$ and hence

$$\frac{1}{3}\, J_T \leq J_{\widetilde{T}} \leq 2\, J_T\,. \tag{3.14}$$

Now (3.9) and (3.10) easily follow using the formula of change of variables in multiple integrals. $\square$

Lemma 3.4. *Let $T \in \mathcal{B}_h$ with $h \in (0, h_0)$ and let $\widetilde{T} \in \widetilde{\mathcal{B}}_h$ be the corresponding modified element. Let $v \in H^1(\operatorname{int} T)$, $\widehat{v} = v \circ (F_T|_{\widehat{T}})$ and $\widetilde{v} = \widehat{v} \circ (F_{\widetilde{T}}^{-1}|_{\widetilde{T}})$. Then*

$$\left|\|v\|^2_{0,T} - \|\widetilde{v}\|^2_{0,\widetilde{T}}\right| \leq c_6\, h_T^{N+1}\, \|\widehat{v}\|^2_{0,\widehat{T}}\,, \tag{3.15}$$

$$\left||v|^2_{1,T} - |\widetilde{v}|^2_{1,\widetilde{T}}\right| \leq c_7\, h_T^{N-1}\, |\widehat{v}|^2_{1,\widehat{T}}\,, \tag{3.16}$$

where the constants c_6 and c_7 depend only on N, C and σ.

Let also $w \in H^1(\operatorname{int} T)$, $\widehat{w} = w \circ (F_T|_{\widehat{T}})$ and $\widetilde{w} = \widehat{w} \circ (F_{\widetilde{T}}^{-1}|_{\widetilde{T}})$. Then

$$\left|\int_T \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\,\mathrm{d}x - \int_{\widetilde{T}} \frac{\partial \widetilde{v}}{\partial x_i}\frac{\partial \widetilde{w}}{\partial x_j}\,\mathrm{d}x\right| \leq c_8\, h_T^{N-1}\, |\widehat{v}|_{1,\widehat{T}}\, |\widehat{w}|_{1,\widehat{T}}\,, \quad i,j = 1,\ldots,N\,, \tag{3.17}$$

where the constant c_8 again depends only on N, C and σ.

Proof. Using the Cramer's rule and inequalities of the type (3.13), we obtain

$$\left| \left(\frac{D F_T^{-1}}{D\,x}\right)_{i,j} - \left(\frac{D F_{\widetilde{T}}^{-1}}{D\,x}\right)_{i,j} \right| \le \widetilde{c}(N,C,\sigma)\,, \quad i,j=1,\dots,N\,.$$

Then (3.15)–(3.17) follow using the formula of change of variables in multiple integrals, the triangular and Hölder's inequalities and the relations (3.11)–(3.14). □

Lemma 3.5. *There exists a constant c_9 depending only on N, σ, Ω and $\widehat{W}$ such that for any $h \in (0,h_0)$*

$$\int_{\Gamma\cap\widetilde{T}} \widetilde{v}^2\,\mathrm{d}\sigma \le c_9\, h_T^{N-1}\,|\widehat{v}_T|^2_{1,\widehat{T}} \quad \forall\, \widetilde{T} \in \widetilde{\mathcal{B}}_h\,,\ v \in W_h. \tag{3.18}$$

(The correspondence among v, $\widetilde{v}$ and $\widehat{v}_T$ is given by (2.7) and (2.8).)

Proof. Let $\Gamma \cap \widetilde{T} \neq \emptyset$. Then, according to the assumption (A4), there exists at least one vertex y of $\widetilde{T}$ such that $\widetilde{v}(y) = 0$. Choose $x \in \Gamma \cap \widetilde{T}$. Then, by the Taylor formula, there exists $z = (1-t)\,x + t\,y$ with some $t \in <0,1>$ such that

$$\widetilde{v}(x) = \nabla\,\widetilde{v}(z)\cdot(x-y)\,.$$

Thus, by (3.11), (3.12) and (3.14),

$$|\widetilde{v}(x)|^2 \le |x-y|^2\,N \sum_{i,j=1}^{N} \left|\frac{\partial \widehat{v}_T}{\partial \widehat{x}_j}(F_{\widetilde{T}}^{-1}(z))\right|^2 \left|\left(\frac{D F_{\widetilde{T}}^{-1}}{D\,x}\right)_{j,i}\right|^2 \le \overline{c}(N,\sigma)\,|\widehat{v}_T|^2_{1,\infty,\widehat{T}}\,.$$

The Lipschitz–continuity of $\partial\Omega$ implies the existence of a constant $\widetilde{c} = \widetilde{c}(\Omega)$ such that for any $\widetilde{T} \subset \mathbb{R}^N$

$$\mathrm{meas}_{N-1}(\Gamma \cap \widetilde{T}) \le \widetilde{c}\,h_{\widetilde{T}}^{N-1}$$

and hence we obtain (3.18) using (3.12) and the equivalence of norms on finite–dimensional spaces. □

Lemma 3.6. *There exists a constant $\widehat{c}$ depending only on N, C, σ, Ω, $\widehat{T}_0$ and $\widehat{W}$ such that for any $h \in (0,h_0)$ and any v, $w \in V_h$, the following inequalities hold:*

$$\|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h\setminus\Omega} \le \widehat{c}\,h\,\|\widetilde{v}\|^2_{1,\Omega}\,, \tag{3.19}$$

$$\left| |v|^2_{1,\Omega_h^B} - |\widetilde{v}|^2_{1,\widetilde{\Omega}_h^B} \right| \le \widehat{c}\,h\,|\widetilde{v}|^2_{1,\Omega}\,, \tag{3.20}$$

$$\left| \|v\|^2_{1,\Omega_h^B} - \|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h^B} \right| \le \widehat{c}\,h\,\|\widetilde{v}\|^2_{1,\Omega}\,, \tag{3.21}$$

$$\int_\Gamma \widetilde{v}^2\,\mathrm{d}\sigma \le \widehat{c}\,h\,|\widetilde{v}|^2_{1,\Omega}\,, \tag{3.22}$$

$$\left| \int_{\Omega_h^B} \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\,\mathrm{d}x - \int_{\widetilde{\Omega}_h^B} \frac{\partial \widetilde{v}}{\partial x_i}\frac{\partial \widetilde{w}}{\partial x_j}\,\mathrm{d}x \right| \le \widehat{c}\,h\,|\widetilde{v}|_{1,\Omega}\,|\widetilde{w}|_{1,\Omega}\,, \quad i,j=1,\dots,N\,. \tag{3.23}$$

Proof. Choose $v \in V_h$. According to (3.1) and (3.9), we have

$$\|\widetilde{v}\|^2_{0,\Omega} \ge \sum_{\widetilde{T}\in\widetilde{\mathcal{B}}_h} \|\widetilde{v}\|^2_{0,F_{\widetilde{T}}(\widehat{T}_0)} \ge c_3 \sum_{T\in\mathcal{B}_h} h_T^N\,\|\widehat{v}_T\|^2_{0,\widehat{T}_0}$$

and similarly, according to (3.1) and (3.10), we have

$$|\widetilde{v}|^2_{1,\Omega} \geq \sum_{\widetilde{T}\in\widetilde{\mathcal{B}}_h} |\widetilde{v}|^2_{1,F_{\widetilde{T}}(\widehat{T}_0)} \geq c_4 \sum_{T\in\mathcal{B}_h} h_T^{N-2}\, |\widehat{v}_T|^2_{1,\widehat{T}_0}\,.$$

Thus, (3.4) and (3.5) yield for any measurable set $\widehat{G}\subset\widehat{T}$

$$\sum_{T\in\mathcal{B}_h} h_T^N\, \|\widehat{v}_T\|^2_{0,\widehat{G}} \leq \frac{c_1}{c_3}\,\mathrm{meas}_N(\widehat{G})\, \|\widetilde{v}\|^2_{0,\Omega} \tag{3.24}$$

and

$$\sum_{T\in\mathcal{B}_h} h_T^{N-2}\, |\widehat{v}_T|^2_{1,\widehat{G}} \leq \frac{c_2}{c_4}\,\mathrm{meas}_N(\widehat{G})\, |\widetilde{v}|^2_{1,\Omega}\,. \tag{3.25}$$

Using (3.3), (3.9), (3.24) and (3.2), we obtain

$$\|\widetilde{v}\|^2_{0,\widetilde{\Omega}_h\setminus\Omega} \leq \sum_{\widetilde{T}\in\widetilde{\mathcal{B}}_h} \|\widetilde{v}\|^2_{0,F_{\widetilde{T}}(\widehat{\Sigma}_h)} \leq 2\sum_{T\in\mathcal{B}_h} h_T^N\, \|\widehat{v}_T\|^2_{0,\widehat{\Sigma}_h} \leq \frac{8\,C\,\sigma\,c_1}{c_3}\, h\, \|\widetilde{v}\|^2_{0,\Omega}$$

and, using (3.3), (3.10), (3.25) and (3.2), we obtain

$$|\widetilde{v}|^2_{1,\widetilde{\Omega}_h\setminus\Omega} \leq \sum_{\widetilde{T}\in\widetilde{\mathcal{B}}_h} |\widetilde{v}|^2_{1,F_{\widetilde{T}}(\widehat{\Sigma}_h)} \leq c_5 \sum_{T\in\mathcal{B}_h} h_T^{N-2}\, |\widehat{v}_T|^2_{1,\widehat{\Sigma}_h} \leq \frac{4\,C\,\sigma\,c_2\,c_5}{c_4}\, h\, |\widetilde{v}|^2_{1,\Omega}\,.$$

The inequalities (3.15) and (3.24) imply

$$\left|\|v\|^2_{0,\Omega_h^B} - \|\widetilde{v}\|^2_{0,\widetilde{\Omega}_h^B}\right| \leq \sum_{T\in\mathcal{B}_h} \left|\|v\|^2_{0,T} - \|\widetilde{v}\|^2_{0,\widetilde{T}}\right| \leq c_6 \sum_{T\in\mathcal{B}_h} h_T^{N+1}\, \|\widehat{v}_T\|^2_{0,\widehat{T}} \leq \frac{c_1\,c_6}{2\,c_3}\, h\, \|\widetilde{v}\|^2_{0,\Omega}$$

and the inequalities (3.16) and (3.25) imply

$$\left||v|^2_{1,\Omega_h^B} - |\widetilde{v}|^2_{1,\widetilde{\Omega}_h^B}\right| \leq \sum_{T\in\mathcal{B}_h} \left||v|^2_{1,T} - |\widetilde{v}|^2_{1,\widetilde{T}}\right| \leq c_7 \sum_{T\in\mathcal{B}_h} h_T^{N-1}\, |\widehat{v}_T|^2_{1,\widehat{T}} \leq \frac{c_2\,c_7}{2\,c_4}\, h\, |\widetilde{v}|^2_{1,\Omega}\,.$$

The inequality (3.23) can be obtained similarly using (3.17). Finally, we obtain by (3.18) and (3.25)

$$\int_\Gamma \widetilde{v}^2\,\mathrm{d}\sigma = \sum_{\widetilde{T}\in\widetilde{\mathcal{B}}_h} \int_{\Gamma\cap\widetilde{T}} \widetilde{v}^2\,\mathrm{d}\sigma \leq c_9 \sum_{T\in\mathcal{B}_h} h_T^{N-1} |\widehat{v}_T|^2_{1,\widehat{T}} \leq \frac{c_2\,c_9}{2\,c_4}\, h\, |\widetilde{v}|^2_{1,\Omega}\,.$$

□

The statements of Lemma 3.6 already make it possible to establish easily the boundedness of the constants $K_1(\Omega_h,\Gamma_h)$ and $K_2(\Omega_h,\Gamma_h)$ in the inequalities (1.1) and (1.3), respectively. In order to get also some generalizations of these discrete inequalities, we first prove the following useful lemma.

Lemma 3.7. *There exist a constant $\widehat{c}'$ depending only on N, C, σ, $\widehat{T}_0$ and $\widehat{W}$ and a constant h_0' depending only on h_0, C and σ such that for any $h\in(0,h_0')$*

$$\left|\int_{\Gamma_h} v^2\,\mathrm{d}\sigma - \int_{\Gamma_h} \widetilde{v}^2\,\mathrm{d}\sigma\right| \leq \widehat{c}' h\, \|\widetilde{v}\|^2_{1,\Omega} \qquad \forall\; v\in V_h\,. \tag{3.26}$$

Proof. Denote for any $T^* \in \mathcal{T}_h$

$$\mathcal{U}(T^*) = \{\, T \in \mathcal{T}_h\,,\ T \cap T^* \neq \emptyset \,\}\,, \qquad U(T^*) = \bigcup_{T\in\mathcal{U}(T^*)} T\,.$$

In consequence of the regularity assumption (A2), there exists a positive constant $\widetilde{c}_1 = \widetilde{c}_1(\sigma)$ such that for any $T \in \mathcal{T}_h$

$$\{\, x \in \Omega_h\,,\ \operatorname{dist}(x,T) \leq \widetilde{c}_1 h_T \,\} \subset U(T)\,.$$

Further, there exists a constant $\widetilde{c}_2 = \widetilde{c}_2(\sigma) > 1$ such that $h_T \leq \widetilde{c}_2\, h_{T^*}$ for any T, $T^* \in \mathcal{T}_h$ satisfying $T \in \mathcal{U}(T^*)$. For any $x \in \Omega_h$ we denote by $\widetilde{x}$ the point obtained by moving the point x according to the assumption (A3). Then we obtain for any $h \in (0, \min\{h_0, \widetilde{c}_1/(C\,\widetilde{c}_2^2)\})$ and any $T \in \mathcal{T}_h$ that $x \in T \;\Rightarrow\; \widetilde{x} \in U(T) \cup (\widetilde{T} \setminus \Omega_h)$ and hence also $\widetilde{x} \in T \;\Rightarrow\; x \in U(T)$. In addition, the whole abscissa $x\widetilde{x}$ lies in $U(T)$ and therefore, we obtain using the Taylor formula that for any $T^* \in \mathcal{T}_h$, $\widetilde{x} \in T^*$ and $v \in V_h$

$$|v(\widetilde{x})^2 - \widetilde{v}(\widetilde{x})^2| = |v(\widetilde{x})^2 - v(x)^2| \leq 2\,|x - \widetilde{x}| \max_{T\in\mathcal{U}(T^*)} \|v\|_{0,\infty,T}\, |v|_{1,\infty,T}\,.$$

In view of (3.11), we infer that $|v|_{1,\infty,T} \leq (N!\,\sigma^{N-1}/h_T)\, |\widehat{v}_T|_{1,\infty,\widehat{T}}$ for any $T \in \mathcal{T}_h$ and therefore, using $|x - \widetilde{x}| \leq C\,\widetilde{c}_2^2\, h_{T^*}^2$, we obtain

$$|v(\widetilde{x})^2 - \widetilde{v}(\widetilde{x})^2| \leq \widetilde{c}_3(N,C,\sigma)\, h_{T^*} \max_{T\in\mathcal{U}(T^*)} \|\widehat{v}_T\|_{0,\infty,\widehat{T}}\, |\widehat{v}_T|_{1,\infty,\widehat{T}}\,.$$

Using the equivalence of norms on finite–dimensional spaces, we have

$$\|\widehat{v}_T\|_{0,\infty,\widehat{T}}\, |\widehat{v}_T|_{1,\infty,\widehat{T}} \leq \widetilde{c}_4(\widehat{W})\, \|\widehat{v}_T\|_{0,\widehat{T}}\, |\widehat{v}_T|_{1,\widehat{T}}\,,$$

which implies in view of (3.9) and (3.10)

$$|v(\widetilde{x})^2 - \widetilde{v}(\widetilde{x})^2| \leq \widetilde{c}_5(N,C,\sigma,\widehat{W})\, h_{T^*}^{2-N} \max_{T\in\mathcal{U}(T^*)} \|\widetilde{v}\|^2_{1,\widetilde{T}}\,.$$

Since $\operatorname{card}\mathcal{U}(T) \leq \widetilde{c}_6(\sigma)$ for any $T \in \mathcal{T}_h$, $h > 0$, we obtain

$$\int_{\partial\Omega_h} |v^2 - \widetilde{v}^2|\,\mathrm{d}\sigma \leq \widetilde{c}_7(N,C,\sigma,\widehat{W})\, h\, \|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h}\,.$$

According to (3.19), we have $\|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h} \leq (1 + \widehat{c}\, h_0)\, \|\widetilde{v}\|^2_{1,\Omega}$, which implies (3.26). □

4. DISCRETE FRIEDRICHS' INEQUALITY

Theorem 4.1. *There exist positive constants h_1 and K_1 such that for any $h \in (0, h_1)$*

$$\|v\|_{1,\Omega_h} \leq K_1\, |v|_{1,\Omega_h} \qquad \forall\ v \in W_h\,. \tag{4.1}$$

Proof. Set $h_1 = \min\{h_0, 1/(4\,\widehat{c})\}$ and choose any $h \in (0, h_1)$ and $v \in W_h$. Let $\widetilde{v}$ be the modification of v defined by (2.8). By (2.3), (2.5), (2.6) and (2.9), we have

$$|v|^2_{1,\Omega_h} = |\widetilde{v}|^2_{1,\Omega} + |\widetilde{v}|^2_{1,\widetilde{\Omega}_h\setminus\Omega} + |v|^2_{1,\Omega_h^B} - |\widetilde{v}|^2_{1,\widetilde{\Omega}_h^B} \tag{4.2}$$

and

$$\|v\|^2_{1,\Omega_h} = \|\widetilde{v}\|^2_{1,\Omega} + \|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h\setminus\Omega} + \|v\|^2_{1,\Omega_h^B} - \|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h^B} . \tag{4.3}$$

Thus, we obtain by (4.2), (3.20), (3.22) and (1.2)

$$|v|^2_{1,\Omega_h} \geq |\widetilde{v}|^2_{1,\Omega} - \widehat{c}\, h\, |\widetilde{v}|^2_{1,\Omega} \geq (1 - 2\,\widehat{c}\, h) \left\{ |\widetilde{v}|^2_{1,\Omega} + \int_\Gamma \widetilde{v}^2 \,\mathrm{d}\sigma \right\} \geq \frac{1 - 2\,\widehat{c}\, h}{\widetilde{K}_1(\Omega,\Gamma)} \|\widetilde{v}\|^2_{1,\Omega}$$

and hence, by (4.3), (3.19) and (3.21),

$$\|v\|^2_{1,\Omega_h} \leq (1 + 2\,\widehat{c}\, h)\, \|\widetilde{v}\|^2_{1,\Omega} \leq \widetilde{K}_1(\Omega,\Gamma)\, \frac{1 + 2\,\widehat{c}\, h}{1 - 2\,\widehat{c}\, h}\, |v|^2_{1,\Omega_h} \leq 3\,\widetilde{K}_1(\Omega,\Gamma)\, |v|^2_{1,\Omega_h} .$$

□

Theorem 4.2. *Assume that there exists a constant $\overline{c}$ such that for any $h \in (0, h_0)$*

$$\left| \int_\Gamma \widetilde{v}^2 \,\mathrm{d}\sigma - \int_{\Gamma_h} \widetilde{v}^2 \,\mathrm{d}\sigma \right| \leq \overline{c}\, h\, \|\widetilde{v}\|^2_{1,\Omega} \qquad \forall\ v \in V_h . \tag{4.4}$$

Then there exist positive constants $\widetilde{h}_1$ and $\widetilde{K}_1$ such that for any $h \in (0, \widetilde{h}_1)$

$$\|v\|^2_{1,\Omega_h} \leq \widetilde{K}_1 \left\{ |v|^2_{1,\Omega_h} + \int_{\Gamma_h} v^2 \,\mathrm{d}\sigma \right\} \qquad \forall\ v \in V_h . \tag{4.5}$$

Proof. According to (3.26) and (4.4), there exists a constant $\overline{c}'$ such that

$$\left| \int_\Gamma \widetilde{v}^2 \,\mathrm{d}\sigma - \int_{\Gamma_h} v^2 \,\mathrm{d}\sigma \right| \leq \overline{c}' h\, \|\widetilde{v}\|^2_{1,\Omega} \qquad \forall\ v \in V_h ,\ \ h \in (0, h_0') . \tag{4.6}$$

Set

$$\widetilde{h}_1 = \min\left\{ h_0', \frac{1}{4\,\widehat{c}}, \frac{1}{2\,(\widehat{c} + \overline{c}')\,\widetilde{K}_1(\Omega,\Gamma)} \right\}$$

and choose any $h \in (0, \widetilde{h}_1)$ and $v \in V_h$. Using (4.2), (3.20), (4.6) and (1.2), we obtain

$$|v|^2_{1,\Omega_h} + \int_{\Gamma_h} v^2 \,\mathrm{d}\sigma \geq |\widetilde{v}|^2_{1,\Omega} + \int_\Gamma \widetilde{v}^2 \,\mathrm{d}\sigma - (\widehat{c} + \overline{c}')\, h\, \|\widetilde{v}\|^2_{1,\Omega} \geq$$

$$\geq \frac{1 - (\widehat{c} + \overline{c}')\,\widetilde{K}_1(\Omega,\Gamma)\, h}{\widetilde{K}_1(\Omega,\Gamma)}\, \|\widetilde{v}\|^2_{1,\Omega} .$$

Thus, using (4.3), (3.19) and (3.21), we get

$$\|v\|^2_{1,\Omega_h} \leq (1 + 2\,\widehat{c}\, h)\, \|\widetilde{v}\|^2_{1,\Omega} \leq \frac{(1 + 2\,\widehat{c}\, h)\,\widetilde{K}_1(\Omega,\Gamma)}{1 - (\widehat{c} + \overline{c}')\,\widetilde{K}_1(\Omega,\Gamma)\, h} \left\{ |v|^2_{1,\Omega_h} + \int_{\Gamma_h} v^2 \,\mathrm{d}\sigma \right\} ,$$

which implies (4.5) with $\widetilde{K}_1 = 3\,\widetilde{K}_1(\Omega,\Gamma)$. □

Remark 4.1. The proofs of Theorem 4.1 and Theorem 4.2 show that the constants K_1 and $\widetilde{K}_1$ can be chosen arbitrarily close to $\widetilde{K}_1(\Omega,\Gamma)^{\frac{1}{2}}$ and $\widetilde{K}_1(\Omega,\Gamma)$, respectively, provided h_1 and $\widetilde{h}_1$ are sufficiently small.

Remark 4.2. Let $\partial\Omega$ and $\partial\Gamma$ be piecewise C^2 and let us consider usual triangulations taking into account boundaries of the C^2–parts of $\partial\Omega$ and the boundary of Γ. Then we obtain using similar techniques as in [4] that

$$\left|\int_\Gamma \widetilde{v}^2\,\mathrm{d}\sigma - \int_{\Gamma_h} \widetilde{v}^2\,\mathrm{d}\sigma\right| \le \overline{c}^* h\,\|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h} \qquad \forall\, \widetilde{v} \in H^1(\widetilde{\Omega}_h)\,,$$

where the constant $\overline{c}^*$ is independent of h. Since, by (3.19), $\|\widetilde{v}\|^2_{1,\widetilde{\Omega}_h} \le (1+\widehat{c}\,h_0)\,\|\widetilde{v}\|^2_{1,\Omega}$ for any $v \in V_h$, we obtain (4.4).

Corollary 4.1. *There exist positive constants h_1' and K_1' such that for any $h \in (0, h_1')$*

$$\|v\|_{1,\Omega_h} \le K_1'\,|v|_{1,\Omega_h} \qquad \forall\, v \in H^1(\Omega_h)\,,\ v = 0 \text{ on } \Gamma_h\,.$$

Proof. Let $X_h \subset H^1(\Omega_h)$ consist of continuous elementwise linear functions and denote $X_h' = \{\, v \in X_h\,,\ v = 0 \text{ on } \Gamma_h \,\}$. Using local L^2–projections on reference macroelements, it is possible to define an interpolation operator $r_h' \in \mathcal{L}(L^1(\Omega_h), X_h')$ satisfying for any $v \in H^1(\Omega_h)$ with $v = 0$ on Γ_h

$$\|v - r_h'\,v\|_{0,\Omega_h} \le \widetilde{c}'h\,|v|_{1,\Omega_h}\,, \qquad |v - r_h'\,v|_{1,\Omega_h} \le \widetilde{c}'\,|v|_{1,\Omega_h}\,,$$

where the constant $\widetilde{c}'$ is independent of h (cf. [1]). Using (4.1) with $W_h = X_h'$, we obtain for any $v \in H^1(\Omega_h)$ with $v = 0$ on Γ_h

$$\|v\|_{0,\Omega_h} \le \|v - r_h'\,v\|_{0,\Omega_h} + \|r_h'\,v\|_{0,\Omega_h} \le (\widetilde{c}'\,h_0 + (1+\widetilde{c}')\,K_1)\,|v|_{1,\Omega_h}\,.$$

□

Corollary 4.2. *Assume that there exists a constant $\overline{c}$ such that for any $h \in (0, h_0)$*

$$\left|\int_\Gamma \widetilde{v}^2\,\mathrm{d}\sigma - \int_{\Gamma_h} \widetilde{v}^2\,\mathrm{d}\sigma\right| \le \overline{c}\,h\,\|\widetilde{v}\|^2_{1,\Omega} \quad \forall\, v \in X_h\,, \tag{4.7}$$

where $X_h \subset H^1(\Omega_h)$ consists of continuous elementwise linear functions. Then there exist positive constants $\widetilde{h}_1'$ and $\widetilde{K}_1'$ such that for any $h \in (0, \widetilde{h}_1')$

$$\|v\|^2_{1,\Omega_h} \le \widetilde{K}_1'\left\{|v|^2_{1,\Omega_h} + \int_{\Gamma_h} v^2\,\mathrm{d}\sigma\right\} \qquad \forall\, v \in H^1(\Omega_h)\,. \tag{4.8}$$

Proof. Analogously as in the proof of Corollary 4.1, there exists an operator $r_h \in \mathcal{L}(L^1(\Omega_h), X_h)$ satisfying for any $v \in H^1(\Omega_h)$

$$\|v - r_h\,v\|_{0,\Omega_h} \le \widetilde{c}\,h\,|v|_{1,\Omega_h}\,, \qquad |v - r_h\,v|_{1,\Omega_h} \le \widetilde{c}\,|v|_{1,\Omega_h}\,, \tag{4.9}$$

where the constant $\widetilde{c}$ is independent of h. Using the local form of (4.9) (cf. [1]), we infer that

$$\|v - r_h\,v\|_{0,\partial\Omega_h} \le \widetilde{c}''h^{\frac12}\,|v|_{1,\Omega_h} \quad \forall\, v \in H^1(\Omega_h)\,, \tag{4.10}$$

where the constant $\widetilde{c}''$ is again independent of h. Further, we have $\|v\|_{0,\Gamma} \le \overline{c}_1\,\|v\|_{1,\Omega}$ $\forall\, v \in H^1(\Omega)$ (cf. [5]) and we obtain by (4.7) and (3.26) with $V_h = X_h$ that, for any $v \in X_h$, $\|v\|^2_{0,\Gamma_h} = \|v\|^2_{0,\Gamma_h} - \|\widetilde{v}\|^2_{0,\Gamma} + \|\widetilde{v}\|^2_{0,\Gamma} \le ((\widehat{c}' + \overline{c})\,h_0 + \overline{c}_1^2)\,\|\widetilde{v}\|^2_{1,\Omega}$. Using (3.21), (2.5), (2.6), (2.9) and (2.3), we obtain $(1 - \widehat{c}\,h)\,\|\widetilde{v}\|^2_{1,\Omega} \le \|v\|^2_{1,\Omega_h}$ and hence, for $h \le 1/(2\,\widehat{c})$, $\|v\|_{0,\Gamma_h} \le \overline{c}_2\,\|v\|_{1,\Omega_h}$ for any $v \in X_h$. Thus, applying (4.9) and (4.10), we get

$$\|v\|_{0,\Gamma_h} \le \overline{c}_3\,\|v\|_{1,\Omega_h} \qquad \forall\ v \in H^1(\Omega_h)\,. \tag{4.11}$$

Now, we obtain by (4.9)–(4.11) for any $v \in H^1(\Omega_h)$

$$\int_{\Gamma_h} (r_h\,v)^2 - v^2\,\mathrm{d}\sigma \le \|r_h\,v - v\|_{0,\Gamma_h}\,\|r_h\,v + v\|_{0,\Gamma_h} \le \overline{c}_4\,h^{\frac{1}{2}}\,\|v\|^2_{1,\Omega_h}$$

and therefore, applying (4.5) with $V_h = X_h$, we have

$$\|r_h\,v\|^2_{0,\Omega_h} \le \widetilde{K}_1\,\left\{\left((1+\widetilde{c})^2 + \overline{c}_4\,h_0^{\frac{1}{2}}\right)\,|v|^2_{1,\Omega_h} + \overline{c}_4\,h^{\frac{1}{2}}\,\|v\|^2_{0,\Omega_h} + \int_{\Gamma_h} v^2\,\mathrm{d}\sigma\right\}\,.$$

Since, according to (4.9), $\|v\|_{0,\Omega_h} \le \widetilde{c}\,h\,|v|_{1,\Omega_h} + \|r_h\,v\|_{0,\Omega}$, we get

$$\|v\|^2_{0,\Omega_h} \le \overline{c}_5\,\left\{|v|^2_{1,\Omega_h} + \int_{\Gamma_h} v^2\,\mathrm{d}\sigma\right\} + \overline{c}_5\,h^{\frac{1}{2}}\,\|v\|^2_{0,\Omega_h}\,.$$

For $h \le 1/(4\,\overline{c}_5^2)$, we obtain (4.8) with $\widetilde{K}_1' = 1 + 2\,\overline{c}_5$. □

5. DISCRETE KORN'S INEQUALITY

Theorem 5.1. *There exist positive constants h_2 and K_2 such that for any $h \in (0, h_2)$*

$$\|\boldsymbol{v}\|^2_{1,\Omega_h} \le K_2 \int_{\Omega_h} (\nabla\,\boldsymbol{v} + \nabla\,\boldsymbol{v}^{\mathrm{T}}) \cdot (\nabla\,\boldsymbol{v} + \nabla\,\boldsymbol{v}^{\mathrm{T}})\,\mathrm{d}x \qquad \forall\ \boldsymbol{v} \in [W_h]^N\,.$$

Proof. For simplicity we denote for any domain $G \subset \mathbb{R}^N$ with a Lipschitz–continuous boundary and for any $\boldsymbol{u} \in H^1(G)^N$

$$|||\boldsymbol{u}|||^2_G = \int_G (\nabla\,\boldsymbol{u} + \nabla\,\boldsymbol{u}^{\mathrm{T}}) \cdot (\nabla\,\boldsymbol{u} + \nabla\,\boldsymbol{u}^{\mathrm{T}})\,\mathrm{d}x\,.$$

Then the following ("continuous") Korn's inequality holds (cf. [6])

$$\|\boldsymbol{u}\|^2_{1,\Omega} \le \widetilde{K}_2(\Omega,\Gamma)\,\left\{|||\boldsymbol{u}|||^2_\Omega + \int_\Gamma |\boldsymbol{u}|^2\,\mathrm{d}\sigma\right\} \qquad \forall\ \boldsymbol{u} \in [H^1(\Omega)]^N\,. \tag{5.1}$$

Now set

$$h_2 = \min\left\{h_0, \frac{1}{4\,\widehat{c}}, \frac{1}{2\,(2\,N+3)\,\widetilde{K}_2(\Omega,\Gamma)\,\widehat{c}}\right\}$$

and consider any $h \in (0, h_2)$ and $\boldsymbol{v} \in [W_h]^N$. Let $\widetilde{\boldsymbol{v}}$ be the modification of $\boldsymbol{v}$ defined by (2.8). By (2.3), (2.5),(2.6) and (2.9), we again have

$$|||\boldsymbol{v}|||^2_{\Omega_h} = |||\widetilde{\boldsymbol{v}}|||^2_\Omega + |||\widetilde{\boldsymbol{v}}|||^2_{\Omega_h\setminus\Omega} + |||\boldsymbol{v}|||^2_{\Omega_h^B} - |||\widetilde{\boldsymbol{v}}|||^2_{\widetilde{\Omega}_h^B}\,. \tag{5.2}$$

Since the relations (4.3) and (3.19)–(3.22) also clearly hold for functions belonging to $[W_h]^N$ and since, by (3.20) and (3.23),

$$\Big||||\boldsymbol{v}|||^2_{\Omega_h^B} - |||\widetilde{\boldsymbol{v}}|||^2_{\widetilde{\Omega}_h^B}\Big| \le 2\,\Big||\boldsymbol{v}|^2_{1,\Omega_h^B} - |\widetilde{\boldsymbol{v}}|^2_{1,\widetilde{\Omega}_h^B}\Big| +$$
$$+2\,\Big|\int_{\Omega_h^B} \nabla\,\boldsymbol{v}\cdot\nabla\,\boldsymbol{v}^{\mathrm{T}}\,\mathrm{d}x - \int_{\widetilde{\Omega}_h^B} \nabla\,\widetilde{\boldsymbol{v}}\cdot\nabla\,\widetilde{\boldsymbol{v}}^{\mathrm{T}}\,\mathrm{d}x\Big| \le 2\,(N+1)\,\widehat{c}\,h\,|\widetilde{\boldsymbol{v}}|^2_{1,\Omega}\,,$$

we obtain by (5.2), (3.22) and (5.1)

$$|||\boldsymbol{v}|||_{\Omega_h}^2 \geq |||\widetilde{\boldsymbol{v}}|||_\Omega^2 - 2\,(N+1)\,\widehat{c}\,h\,|\widetilde{\boldsymbol{v}}|_{1,\Omega}^2 \geq$$
$$\geq |||\widetilde{\boldsymbol{v}}|||_\Omega^2 + \int_\Gamma |\widetilde{\boldsymbol{v}}|^2\,\mathrm{d}\sigma - (2\,N+3)\,\widehat{c}\,h\,\|\widetilde{\boldsymbol{v}}\|_{1,\Omega}^2 \geq \frac{1-(2\,N+3)\,\widetilde{K}_2(\Omega,\Gamma)\,\widehat{c}\,h}{\widetilde{K}_2(\Omega,\Gamma)}\,\|\widetilde{\boldsymbol{v}}\|_{1,\Omega}^2$$

and hence, by (4.3), (3.19) and (3.21),

$$\|\boldsymbol{v}\|_{1,\Omega_h}^2 \leq (1+2\,\widehat{c}\,h)\,\|\widetilde{\boldsymbol{v}}\|_{1,\Omega}^2 \leq \frac{(1+2\,\widehat{c}\,h)\,\widetilde{K}_2(\Omega,\Gamma)}{1-(2\,N+3)\,\widetilde{K}_2(\Omega,\Gamma)\,\widehat{c}\,h}\,|||\boldsymbol{v}|||_{\Omega_h}^2 \leq 3\,\widetilde{K}_2(\Omega,\Gamma)\,|||\boldsymbol{v}|||_{\Omega_h}^2 .$$

□

Theorem 5.2. *Let the assumption of Theorem 4.2 be satisfied. Then there exist positive constants $\widetilde{h}_2$ and $\widetilde{K}_2$ such that for any $h \in (0, \widetilde{h}_2)$*

$$\|\boldsymbol{v}\|_{1,\Omega_h}^2 \leq \widetilde{K}_2 \left\{ \int_{\Omega_h} (\nabla\,\boldsymbol{v} + \nabla\,\boldsymbol{v}^{\mathrm{T}}) \cdot (\nabla\,\boldsymbol{v} + \nabla\,\boldsymbol{v}^{\mathrm{T}})\,\mathrm{d}x + \int_{\Gamma_h} |\boldsymbol{v}|^2\,\mathrm{d}\sigma \right\} \qquad \forall\ \boldsymbol{v} \in [V_h]^N .$$

Proof. We obtain the statement by modifying the proof of Theorem 5.1. Applying (4.6), we infer that for any $\boldsymbol{v} \in [V_h]^N$

$$|||\boldsymbol{v}|||_{\Omega_h}^2 + \int_{\Gamma_h} |\boldsymbol{v}|^2\,\mathrm{d}\sigma \geq \frac{1-(2\,(N+1)\,\widehat{c} + \overline{c}')\,\widetilde{K}_2(\Omega,\Gamma)\,h}{\widetilde{K}_2(\Omega,\Gamma)}\,\|\widetilde{\boldsymbol{v}}\|_{1,\Omega}^2 .$$

Thus, the theorem follows using $\|\boldsymbol{v}\|_{1,\Omega_h}^2 \leq (1+2\,\widehat{c}\,h)\,\|\widetilde{\boldsymbol{v}}\|_{1,\Omega}^2$. □

Remark 5.1. Again, we see that K_2 and $\widetilde{K}_2$ can be chosen arbitrarily close to $\widetilde{K}_2(\Omega,\Gamma)$ for h_2 and $\widetilde{h}_2$ sufficiently small.

6. VALIDITY OF THE ASSUMPTION (A3)

It can be expected that the assumption (A3) holds for a lot of usual domains used in applications. Here, for simplicity, we prove the validity of this assumption only for domains with a C^2–boundary, although a piecewise C^2–boundary would be sufficient.

Recall that a bounded domain $\Omega \subset \mathbb{R}^N$ has a C^2–boundary $\partial\Omega$ if there exist positive real numbers a, b such that for each $x^0 \in \partial\Omega$, the Cartesian coordinate system can be rotated and translated to x^0 in such a way that there exists $f \in C^2(<\!-a,a\!>^{N-1})$ such that for any $\overline{x} \in (-a,a)^{N-1}$

$$\begin{aligned}
&(\overline{x}, f(\overline{x})) \in \partial\Omega\,, \\
&f(\overline{x}) < x_N < f(\overline{x}) + b \quad \Rightarrow \quad (\overline{x}, x_N) \in \Omega\,, \\
&f(\overline{x}) - b < x_N < f(\overline{x}) \quad \Rightarrow \quad (\overline{x}, x_N) \notin \overline{\Omega}\,.
\end{aligned}$$

Thus we can choose a finite number M of local coordinate systems and functions $f_1, \ldots, f_M \in C^2(<\!-a,a\!>^{N-1})$ such that $\partial\Omega$ is covered by the graphs $\{(\overline{x}, f_i(\overline{x})),\ \overline{x} \in (-a/4, a/4)^{N-1}\}$, $i = 1, \ldots, M$, and we define

$$C_1 = \max_{i=1,\ldots,M} |f_i|_{1,\infty,(-a,a)^{N-1}}\,, \qquad C_2 = \max_{i=1,\ldots,M} |f_i|_{2,\infty,(-a,a)^{N-1}}\,.$$

Further, for a given triangulation $\mathcal{T}_h$, we denote by T' any edge (if $N = 2$) or any face (if $N = 3$) belonging to $\partial\Omega_h$. If $N = 2$, we introduce the abscissa $K(T', r)$ lying in the line determined by T' and having a length $2\,r$ and a middle in the middle of T'. Similarly, if $N = 3$, we introduce the circle $K(T', r)$ which lies in the plane determined by T' and has a radius r and a centre in the barycentre of T'.

Lemma 6.1. *For any $\alpha_0 > 0$ there exists $h_3 = h_3(\alpha_0, \sigma, C_1, C_2) > 0$ such that for any $h \in (0, h_3)$ and any $T' \in \partial\Omega_h$, the angle between the outward normal vector to $\partial\Omega_h$ on T' and the outward normal vector to $\partial\Omega$ in any vertex of T' is at most α_0.*

Proof. The proof is straightforward and is based on a suitable application of the Taylor formula. □

Lemma 6.2. *There exist $h_4 = h_4(\sigma, C_1, C_2) > 0$ and $C_3 = C_3(\sigma, C_2) > 0$ such that for any $h \in (0, h_4)$, the following statement holds: Choose any $x \in K(T', r)$, where $T' \subset T \cap \partial\Omega_h$, $T \in \mathcal{B}_h$ and $r \in (h_T, 2\,h)$. Let $y \in \partial\Omega$ be the nearest point of the boundary $\partial\Omega$ satisfying $(x - y) \perp T'$. Then*

$$|x - y| \leq C_3\, r^2\,. \tag{6.1}$$

Proof. It follows from Lemma 6.1 that for h_4 sufficiently small and any $x \in K(T', r)$, there always exists $y \in \partial\Omega$ satisfying $(x - y) \perp T'$. Then the inequality (6.1) can be obtained using the Taylor formula. □

Lemma 6.3. *There exists a number $\varepsilon = \varepsilon(\sigma)$ such that, for any triangulation $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$, the lengths l_1, l_2 of any two edges containing a common vertex satisfy*

$$\frac{1}{\varepsilon}\, l_2 \leq l_1 \leq \varepsilon\, l_2\,. \tag{6.2}$$

Proof. It can be easily shown that there exists an integer $m = m(\sigma)$ such that, for any triangulation $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$, there exist at most m elements $T \in \mathcal{T}_h$ containing the same vertex. Hence, (6.2) holds with $\varepsilon = \sigma^m$. □

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a bounded domain with a C^2–boundary and let the assumption (A2) be satisfied. Then the assumption (A3) is valid.*

Proof. *1. The two–dimensional case.* Choose $\alpha_0 > 0$ satisfying $2\,\alpha_0 + \operatorname{arctg} C_1 < \pi/2$. We shall show that there exists $h_5 < \min\{h_3(\alpha_0, \sigma, C_1, C_2), h_4(\sigma, C_1, C_2), a/4\}$ such that for any triangulation $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ with $h \in (0, h_5)$, a modified triangulation $\tilde{\mathcal{T}}_h$ can be created by moving any vertex $x \in \partial\Omega_h$ the distance $d_x = C_3\, h_x^2 / \cos\alpha_0$ in the direction of the outward normal vector to $\partial\Omega$. Here, h_x is the length of the longest edge lying on $\partial\Omega_h$ and containing x. Hence, by Lemma 6.3, $d_x \leq (C_3\, \varepsilon^2 / \cos\alpha_0)\, h_T^2$ for any $T \in \mathcal{T}_h$ containing the vertex x. Now, consider any edge $l \equiv T' \subset \partial\Omega_h$ and choose a local coordinate system where a part of l is described on the interval $(-a/4, a/4)$. The situation is depicted, for h_5 sufficiently small, in Fig. 2. The quadrilateral ABCD demarcates the region under AD where points of $\partial\Omega$ can appear and the quadrilateral EFGH demarcates an analogical region for the abscissa EH (cf. Lemma 6.2). Moreover, we obtain by Lemma 6.1 and

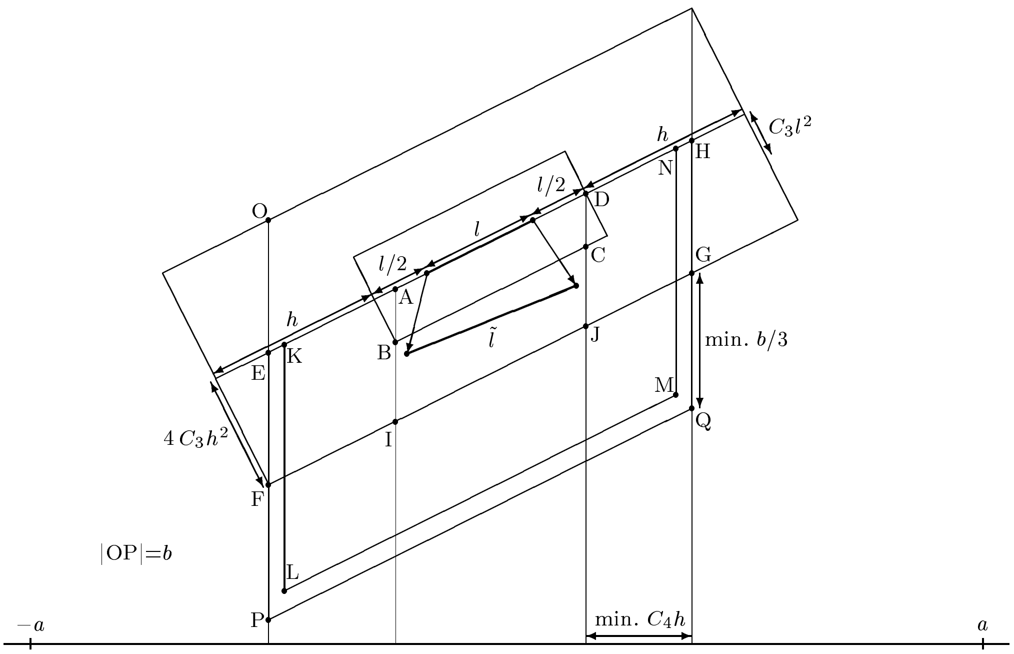

Figure 2. Displacement of a boundary edge.

Lemma 6.3 that, for h_5 sufficiently small, the moved edge $\tilde{l}$ lies in the quadrilateral BIJC (it lies between EH and FG since $d_x \le \sqrt{2}\, C_3\, h^2$), the displacement of any vertex x is at most one third of the length of any boundary edge containing x (so that no collision among moved edges described in the same local coordinate system can appear and the vertex order is preserved for any element T) and points of any moved edge described in another local coordinate system cannot appear in the quadrilateral KLMN (since $|\mathrm{OP}| = b$, we have $\partial\Omega \cap \mathrm{FPQG} = \emptyset$) so that any collision with such edges is excluded as well. Thus, $\widetilde{\mathcal{T}}_h$ really is a triangulation of $\widetilde{\Omega}_h$ and the inclusions (2.3) are satisfied. Since there are bounds given by (2.1) for the geometrical form of any element T, we also obtain (2.4).

2. The three–dimensional case. We can proceed in a very similar fashion as in the two–dimensional case. The only mentionable difference consists in replacing the abscissae on the line determined by l (cf. Fig. 2) by suitable rings lying in the plane determined by the triangle $T' \subset \partial\Omega_h$. Fig. 2 then corresponds to a vertical cut through this three–dimensional arrangement. Again, the conditions (2.1) and (6.1) ensure the validity of (2.4). □

For domains, the boundary of which is only piecewise C^2 (and Lipschitz–continuous, of course), the normal vector to $\partial\Omega$ does not exist in some points in general and therefore a special choice of the directions of vertex diplacements has to be made. However, in view of the Lipschitz–continuity of $\partial\Omega$, that does not present any problems. For example, we can use the cone property, which says that there exists a finite cone $B \subset \mathbb{R}^N$ such that, for any $x_0 \in \partial\Omega$, the cone B can be translated and rotated onto $B_{x_0} \subset \mathbb{R}^N \setminus \Omega$ having the vertex x_0. Then the axis of a suitable cone B_{x_0} can serve as the direction of vertex displacements in x_0. Thus, for usual triangulations taking into account the boundaries of the C^2–parts of $\partial\Omega$, we again obtain the validity of (A3).

Remark 6.1. It is easy to construct domains Ω which have a Lipschitz–continuous boundary only and satisfy (A3) for suitable families of triangulations. For example, let $\Omega = \operatorname{int}\operatorname{conv}\{(1/n, 1/n^2)\,,\ n \in \mathrm{Z} \setminus \{0\}\}$ be the interior of the convex hull of the points $(1/n, 1/n^2)$. Then the assumption (A3) is satisfied for triangulations $\mathcal{T}_h$ such that, for $h \in (0, 0.5)$, the points $(1/n, 1/n^2)$ with $0 < |n| \leq 1 + 1/h$ belong to the set of vertices of $\mathcal{T}_h$ and $h \leq \sigma' h_T$ for $T \in \mathcal{T}_h$ containing boundary vertexes (x_1, x_2) with $x_2 \leq h^2$.

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