Function spaces

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain, i.e., a bounded open and connected set. We denote by $C(\Omega)$ the set of continuous real functions defined in Ω. The notation $C^k(\Omega)$ with $k \in \mathbb{N}$ denotes the subset of $C(\Omega)$ consisting of functions having continuous derivatives up to the order k. For a function $v \in C^k(\Omega)$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_1, \dots, \alpha_d \in \mathbb{N}_0$ and $|\alpha| := \alpha_1 + \dots + \alpha_d \leq k$, we denote by $D^\alpha v$ the derivative

$$\frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} .$$

The set $C^\infty(\Omega)$ consists of functions which belong to $C^k(\Omega)$ for all $k \in \mathbb{N}$. We set

$$C_0^\infty(\Omega) = \{v \in C^\infty(\Omega) \, ; \; \operatorname{supp} v \subset \Omega\} ,$$

where

$$\operatorname{supp} v = \overline{\{x \in \Omega \, ; \; v(x) \neq 0\}} .$$

For $x \in \mathbb{R}^d$ and a multi-index α, we define

$$x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d} .$$

Then

$$P_k = \left\{ \sum_{|\alpha| \leq k} \gamma_\alpha \, x^\alpha \, ; \; \gamma_\alpha \in \mathbb{R} \right\}$$

is the space of polynomials up to the degree $k \geq 0$ and

$$Q_k = \left\{ \sum_{0 \leq \alpha_i \leq k, \, i=1,\dots,d} \gamma_\alpha \, x^\alpha \, ; \; \gamma_\alpha \in \mathbb{R} \right\}$$

is the space of polynomials up to the degree $k \geq 0$ in each variable. For any set $A \subset \mathbb{R}^d$, we denote by $P_k(A)$ the space of functions from P_k restricted onto A. The notation $Q_k(A)$ has an analogous meaning.

We denote by $L^p(\Omega)$ with $p \in [1, \infty)$ the *Lebesgue space*

$$L^p(\Omega) = \left\{ v : \Omega \to \mathbb{R} \, ; \; v \text{ is measurable}, \; \int_\Omega |v(x)|^p \, \mathrm{d}x < \infty \right\} .$$

It is a Banach space with respect to the norm

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p \, \mathrm{d}x \right)^{1/p} .$$

For $p = 2$, it is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(x) \, v(x) \, \mathrm{d}x .$$

Furthermore, given a measurable set $G \subset \mathbb{R}^d$, we shall use the notation

$$(u, v)_G = \int_G u(x) \, v(x) \, \mathrm{d}x .$$

For $p = \infty$, we set

$$L^\infty(\Omega) = \left\{ v : \Omega \to \mathbb{R} \, ; \; v \text{ is measurable}, \; \operatorname*{ess\,sup}_{x \in \Omega} |v(x)| < \infty \right\} .$$

It is a Banach space with respect to the norm

$$\|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|\,.$$

We recall that the essential supremum is defined by

$$\operatorname*{ess\,sup}_{x\in\Omega} |v(x)| = \inf_{G\in\mathscr{M}_0(\Omega)} \sup_{x\in\Omega\setminus G} |v(x)|\,,$$

where $\mathscr{M}_0(\Omega)$ is the set of measurable subsets of Ω with zero measure. Finally, we define the space of locally integrable functions

$$L^{1,loc}(\Omega) = \left\{ v : \Omega \to \mathbb{R}\,;\ v \text{ is measurable}, \ \int_K |v(x)|\,\mathrm{d}x < \infty \ \ \forall\, K \subset \Omega \text{ compact} \right\}.$$

Obviously, $L^p(\Omega) \subset L^{1,loc}(\Omega)$ for any $p \in [1,\infty]$.

Functions from the space $L^{1,loc}(\Omega)$ do not possess classical derivatives in general, however, if we interpret them as distributions, derivatives of arbitrary orders can be defined. These derivatives are again distributions. If such a derivative can be identified with a function from $L^{1,loc}(\Omega)$, we call it *weak derivative*. More precisely, given a multi-index α, a function $v_\alpha \in L^{1,loc}(\Omega)$ is the α-th weak derivative of a function $v \in L^{1,loc}(\Omega)$ if

$$\int_\Omega v_\alpha\, \varphi\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\, D^\alpha \varphi\,\mathrm{d}x \qquad \forall\, \varphi \in C_0^\infty(\Omega)\,.$$

If the weak derivative exists, than it is determined uniquely (as an element of $L^{1,loc}(\Omega)$). Note that a function $v \in C^k(\Omega)$ with $k \in \mathbb{N}$ satisfies

$$\int_\Omega (D^\alpha v)\, \varphi\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\, D^\alpha \varphi\,\mathrm{d}x \qquad \forall\, \varphi \in C_0^\infty(\Omega)$$

for any multi-index α with $|\alpha| \leq k$. This shows that whenever the classical derivatives exist, they coincide with the weak derivatives. Therefore, we shall use the notation $D^\alpha v$ also for weak derivatives in the following.

Given $p \in [1,\infty]$ and $k \in \mathbb{N}_0$, the *Sobolev space* $W^{k,p}(\Omega)$ is the set

$$W^{k,p}(\Omega) = \{ v \in L^p(\Omega)\,;\ D^\alpha v \in L^p(\Omega)\ \forall\, |\alpha| \leq k \}\,,$$

i.e., it is the set of all functions from $L^p(\Omega)$ whose weak derivatives up to the order k exist and belong to $L^p(\Omega)$. The Sobolev space $W^{k,p}(\Omega)$ is a Banach space with respect to the norm

$$\|v\|_{k,p,\Omega} = \begin{cases} \left(\displaystyle\sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } \ p \in [1,\infty)\,, \\[2ex] \displaystyle\max_{|\alpha|\leq k} \|D^\alpha v\|_{L^\infty(\Omega)} & \text{if } \ p = \infty\,. \end{cases}$$

Furthermore, we introduce the seminorms

$$|v|_{k,p,\Omega} = \begin{cases} \left(\displaystyle\sum_{|\alpha|= k} \|D^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } \ p \in [1,\infty)\,, \\[2ex] \displaystyle\max_{|\alpha|= k} \|D^\alpha v\|_{L^\infty(\Omega)} & \text{if } \ p = \infty\,. \end{cases}$$

Note that $W^{0,p}(\Omega) = L^p(\Omega)$ and $\|\cdot\|_{0,p,\Omega} = \|\cdot\|_{L^p(\Omega)}$. The spaces $W^{k,p}(\Omega)$ are separable for $p \in [1,\infty)$ and reflexive for $p \in (1,\infty)$. For $p = 2$, the spaces $W^{k,2}(\Omega)$ are Hilbert spaces with the inner product

$$(u,v)_{k,\Omega} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)\,.$$

The space $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$. The space of continuous linear functionals on the space $W^{k,p}(\Omega)$ will be denoted by $[W^{k,p}(\Omega)]'$.

For $p = 2$, we shall drop the index p and denote the Sobolev spaces by H instead of W. Thus, we shall use the notation

$$\begin{aligned} H^k(\Omega) &= W^{k,2}(\Omega)\,, \qquad & H_0^k(\Omega) &= W_0^{k,2}(\Omega)\,,\\ \|\cdot\|_{k,\Omega} &= \|\cdot\|_{k,2,\Omega}\,, \qquad & |\cdot|_{k,\Omega} &= |\cdot|_{k,2,\Omega}\,. \end{aligned}$$

It can be shown that the there exists a constant C depending only on Ω, k and p such that

$$\|v\|_{p,k,\Omega} \le C\,|v|_{p,k,\Omega} \qquad \forall\ v \in W_0^{k,p}(\Omega)\,.$$

This statement is known as Friedrichs' inequality.

The Friedrichs inequality holds for any measurable domain $\Omega \subset \mathbb{R}^d$. However, the proofs of many other properties of Sobolev spaces require some assumptions on the regularity of the boundary $\partial\Omega$ of Ω. Here, we shall use the following concept.

Definition 0.1. *A bounded domain $\Omega \subset \mathbb{R}^d$ $(d \ge 2)$ is of class $C^{k,1}$ (with $k \in \mathbb{N}_0$) if there exists a finite number of local coordinate systems $S_1,\dots,S_M$ and functions $f_1,\dots,f_M$ and if there exist numbers $a, b > 0$ such that*

- *the functions $f_1,\dots,f_M$ are k-times continuously differentiable and have Lipschitz-continuous derivatives of order k on the closure of the set*

$$K_{d-1} = \{y = (y_1,\dots,y_{d-1})\,;\ |y_j| < a,\ j = 1,\dots,d-1\}\,;$$

- *for any $x \in \partial\Omega$, there exist $i \in \{1,\dots,M\}$ and $y \in K_{d-1}$ such that $x = (y, f_i(y))$ in the local coordinate system S_i;*
- *in each local coordinate system S_i, $i = 1,\dots,M$,*

$$\begin{aligned} (y,y_d) \in \Omega \quad &\text{if} \quad y \in \overline{K_{d-1}}\,,\ f_i(y) < y_d < f_i(y) + b\,,\\ (y,y_d) \notin \overline{\Omega} \quad &\text{if} \quad y \in \overline{K_{d-1}}\,,\ f_i(y) - b < y_d < f_i(y)\,. \end{aligned}$$

If Ω is of class $C^{0,1}$, we say that Ω has a Lipschitz-continuous boundary.

Remark 0.1. *Bounded convex domains in $\mathbb{R}^d$ have Lipschitz-continuous boundaries. A bounded polygonal domain in $\mathbb{R}^2$ has a Lipschitz-continuous boundary if the boundary represents a simple curve. This is not satisfied, e.g., for the domain $(0,3)^2 \setminus \{[1,2]^2 \cup [0,1]^2\}$ whose boundary is not Lipschitz-continuous at the point $(1,1)$. A bounded polyhedral domain in $\mathbb{R}^3$ has not a Lipschitz-continuous boundary in general. An example is the interior of the set $[0,1]\times[0,2]\times[0,1] \cup [0,2]\times[1,2]\times[1,2]$ whose boundary is not Lipschitz-continuous at the point $(1,1,1)$.*

An example of statements that require a certain regularity of the boundary are imbedding theorems. If X and Y are two normed linear spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, we say that X is continuously imbedded in Y and write $X \hookrightarrow Y$ if $X \subset Y$ and the identity mapping $i: X \to Y$ is continuous (i.e., there is a constant C such that $\|x\|_Y \le C\,\|x\|_X$ for any $x \in X$). We say that X is compactly imbedded in Y and write $X \hookrightarrow\hookrightarrow Y$ if $X \subset Y$ and the identity mapping $i: X \to Y$ is compact.

Theorem 0.1. (Sobolev imbedding theorem) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz-continuous boundary. Consider any $k, j \in \mathbb{N}_0$ and $p, r \in [1, \infty)$. Then*

$$W^{k,p}(\Omega) \hookrightarrow W^{j,r}(\Omega) \qquad \text{if} \qquad 0 \le j \le k \quad \text{and} \quad \frac{1}{p} - \frac{k-j}{d} \le \frac{1}{r}\,,$$

$$W^{k,p}(\Omega) \hookrightarrow C^j(\overline{\Omega}) \qquad \text{if} \qquad \frac{1}{p} - \frac{k-j}{d} < 0\,.$$

Theorem 0.2. (Rellich, Kondrasov) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz-continuous boundary. Then*

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{k-1,p}(\Omega) \qquad \forall\ k \in \mathbb{N},\ p \in [1, \infty]\,.$$

Finally, let us mention that, for any function space X, we shall denote by X^d the space of vector-valued functions with d components, each of them belongs to X.