The chain relation in sofic subshifts

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- Shifts and subshifts
- The chain relation
- 2 Characterisation of the chain relation
 - Linking graph
 - Theorem about chain relation
 - Corollaries

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Shifts and subshifts The chain relation

Basics

• We are interested in the structure of biinfinite words $A^{\mathbb{Z}}$.

We can equip A^ℤ with a metric *ρ*; the distance *ρ*(*x*, *y*) of *x* ≠ *y* is equal to 2^{−n} where *n* is the absolute value of the first index where *x* differs from *y*.

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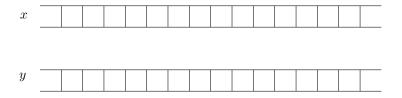
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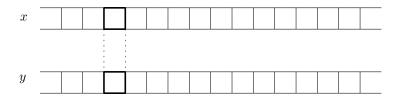


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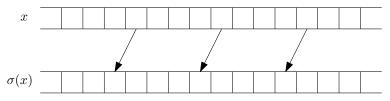
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Shifts and subshifts The chain relation

Basics, cont.

• Define the *shift map* by $\sigma(x)_i = x_{i+1}$.



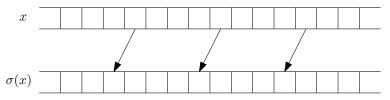
• Sofic subshift is a set $\Sigma \subseteq A^{\mathbb{Z}}$ that can be described by a labelled graph.

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Shifts and subshifts The chain relation

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Shifts and subshifts The chain relation

Labelled graph

- *Labelled graph* is an oriented multidigraph whose edges are labelled by letters from *A*.
- $x \in \Sigma$ iff x has a *presentation* in G: There exists a biinfinite walk in G labelled by letters from x.
- Without loss of generality assume that *G* is *essential*, that is every vertex has at least one outgoing an at least one ingoing edge.

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- An ε-chain from the word x to the word y is sequence of words x⁰, x¹,..., xⁿ ∈ Σ such that x⁰ = x, xⁿ = y and ρ(σ(xⁱ), xⁱ⁺¹) < ε.
- The words x, y ∈ Σ are in the *chain relation* C if for every ε > 0 there exists an ε-chain (of nonzero length) from x to y.

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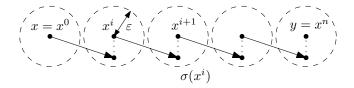
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ε -chain in picture



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Shifts and subshifts The chain relation

ε -chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε -chain:

...aab aa... ..aabaa...aa aa... Hop! ...aa...a a...aabaa... ...aa baa...

We have managed to shift the word to the right.

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How to describe the chain relation in a general sofic subshift Σ?

- The main idea: We can jump between some vertices of G.
- Call such pairs of vertices linked.

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Linking graph Theorem about chain relation Corollaries

Linked vertices

- Call two vertices u, v of a labelled graph G linked if for any length n there exists a word w of length n that has presentations beginning in both u, v and not leaving the components of u, v.
- By joining all pairs of linked vertices we obtain the *linking* graph G/_≈.
- We have a natural projection of *G* onto $G/_{\approx}$.

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Linking graph Theorem about chain relation Corollaries

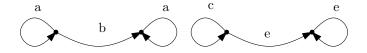
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Linking graph Theorem about chain relation Corollaries

Linked vertices in picture

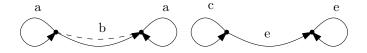


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Linking graph Theorem about chain relation Corollaries

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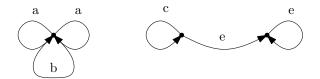


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Linking graph Theorem about chain relation Corollaries

Linking graph in picture



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Linking graph Theorem about chain relation Corollaries

Components of $G/_{\approx}$

- The components of the graph G/_≈ can be partially ordered by the relation K ≤ L meaning "there exists a walk from K to L".
- For x infinite word define α(x) and ω(x) as the components of G/_≈ where the image of a presentation of x begins and ends.
- The components α(x), ω(x) are well-defined: They always exist and do not depend on the choice of presentation of x.

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Linking graph Theorem about chain relation Corollaries

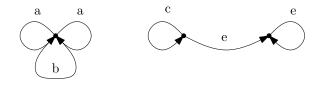
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Linking graph Theorem about chain relation Corollaries

The components of $G/_{\approx}$ in picture



 K_1 is incomparable with K_2, K_3 and $K_2 \leq K_3$.

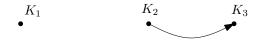
 $\alpha(\boldsymbol{a}^{\infty}) = \omega(\boldsymbol{a}^{\infty}) = K_1$

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Linking graph Theorem about chain relation Corollaries

The components of $G/_{\approx}$ in picture



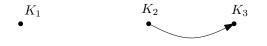
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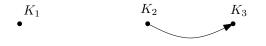
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Linking graph Theorem about chain relation Corollaries

Theorem about chain relation

Theorem

Let Σ be a sofic subshift, G its labelled graph. Let $x, y \in \Sigma$. Then $(x, y) \in C$ iff $\omega(x) \le \alpha(y)$ or $y = \sigma^n(x)$ for some n > 0.

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Linking graph Theorem about chain relation Corollaries

Other properties of subshifts

- A subshift is *chain-transitive* if every two its words are in the chain relation.
- A susbift is *chain-mixing* if for every two words x, y ∈ Σ and ε > 0 exists a k such that for all n > k there exists an ε-chain from x to y of length n.
- Chain transitivity is often used when describing dynamic systems, the chain mixing property can be useful for finding attractors of celluar automata.

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Characterising chain transitive and chain mixing subshifts

Theorem

Let Σ be a sofic subshift, G its essential graph. Then Σ is chain transitive iff $G/_{\approx}$ is (strongly) connected.

Theorem

Let Σ be a sofic subshift, G its essential graph. Then Σ is chain mixing iff $G_{|_{\approx}}$ is (strongly) connected and aperiodic.

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Intermezzo: Aperiodic graphs

A graph is *periodic* iff V(G) can be partitioned into n > 1 disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ such that every edge $e \in E(G)$ leads from some $v \in V_k$ to some $u \in V_{k+1}$ for a suitable k.

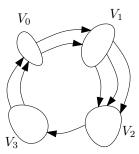
A graph is *aperiodic* if it is not periodic.

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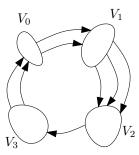


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A graph is aperiodic if it is not periodic.



Informally, an attractor of a dynamic system is a closed set that attracts all trajectories from its neighbourhood.

[heorem]

The attractors of the dynamic system (Σ, σ) are precisely all the subshifts described by the preimages of nonempty terminal subgraphs of G/ $_{\approx}$.

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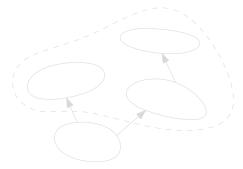
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Intermezzo: Terminal subgraphs

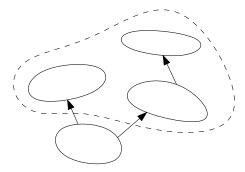
A subgraph *H* of $G/_{\approx}$ is terminal iff there are no edges leading from V(H) to $V(G) \setminus V(H)$.



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Summary

• Using the linking graph we can describe the chain relation in sofic subshifts.

• Using this knowledge we can explicitely describe:

- Chain transitivity
- The chain-mixing property
- The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

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