The chain relation in sofic subshifts

Alexandr Kazda

Charles University, Prague
alexak@atrey.karlin.mff.cuni.cz

WSDC 2007

Outline

Basics

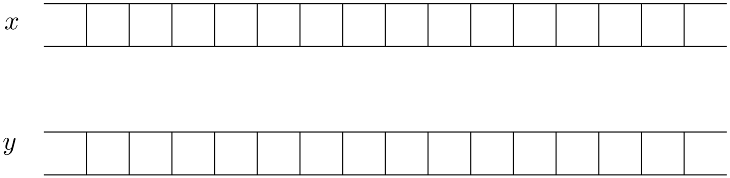
- We are interested in the structure of biinfinite words $A^{\mathbb{Z}}$.
- We can equip $A^{\mathbb{Z}}$ with a metric ϱ; the distance $\varrho(x, y)$ of $x \neq y$ is equal to 2^{-n} where n is the absolute value of the first index where x differs from y.

Basics

- We are interested in the structure of biinfinite words $A^{\mathbb{Z}}$.
- We can equip $A^{\mathbb{Z}}$ with a metric ϱ; the distance $\varrho(x, y)$ of $x \neq y$ is equal to 2^{-n} where n is the absolute value of the first index where x differs from y.

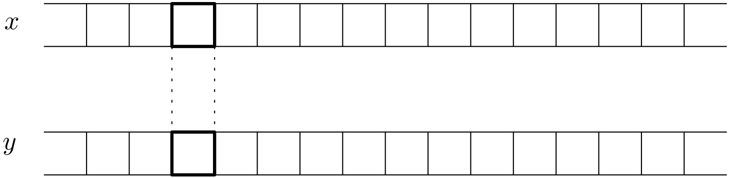
Basics

- We are interested in the structure of biinfinite words $A^{\mathbb{Z}}$.
- We can equip $A^{\mathbb{Z}}$ with a metric ϱ; the distance $\varrho(x, y)$ of $x \neq y$ is equal to 2^{-n} where n is the absolute value of the first index where x differs from y.



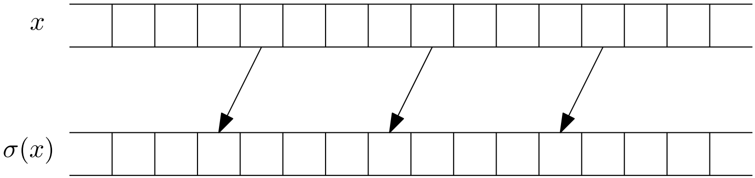
Basics

- We are interested in the structure of biinfinite words $A^{\mathbb{Z}}$.
- We can equip $A^{\mathbb{Z}}$ with a metric ϱ; the distance $\varrho(x, y)$ of $x \neq y$ is equal to 2^{-n} where n is the absolute value of the first index where x differs from y.



Basics, cont.

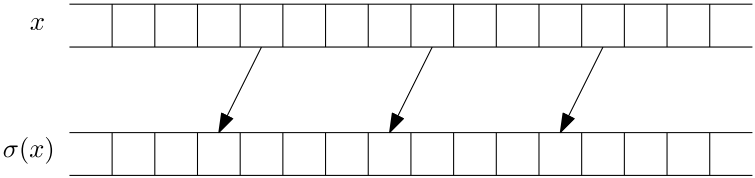
- Define the *shift map* by $\sigma(x)_i = x_{i+1}$.



- *Sofic subshift* is a set $\Sigma \subseteq A^{\mathbb{Z}}$ that can be described by a labelled graph.

Basics, cont.

- Define the *shift map* by $\sigma(x)_i = x_{i+1}$.



- *Sofic subshift* is a set $\Sigma \subseteq A^{\mathbb{Z}}$ that can be described by a labelled graph.

Labelled graph

- *Labelled graph* is an oriented multidigraph whose edges are labelled by letters from A.
- $x \in \Sigma$ iff x has a *presentation* in G: There exists a biinfinite walk in G labelled by letters from x.
- Without loss of generality assume that G is *essential*, that is every vertex has at least one outgoing an at least one ingoing edge.

Labelled graph

- *Labelled graph* is an oriented multidigraph whose edges are labelled by letters from A.
- $x \in \Sigma$ iff x has a *presentation* in G: There exists a biinfinite walk in G labelled by letters from x.
- Without loss of generality assume that G is *essential*, that is every vertex has at least one outgoing an at least one ingoing edge.

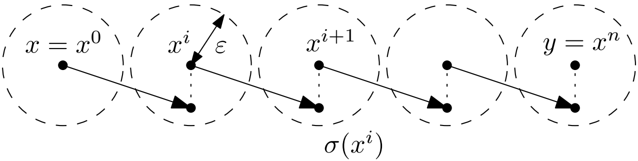
Labelled graph

- *Labelled graph* is an oriented multidigraph whose edges are labelled by letters from A.
- $x \in \Sigma$ iff x has a *presentation* in G: There exists a biinfinite walk in G labelled by letters from x.
- Without loss of generality assume that G is *essential*, that is every vertex has at least one outgoing an at least one ingoing edge.

- An ε-chain from the word x to the word y is sequence of words $x^0, x^1, \ldots, x^n \in \Sigma$ such that $x^0 = x, x^n = y$ and $\rho(\sigma(x^i), x^{i+1}) < \varepsilon$.
- The words $x, y \in \Sigma$ are in the *chain relation* $\mathcal{C}$ if for every $\varepsilon > 0$ there exists an ε-chain (of nonzero length) from x to y.

- An ε-chain from the word x to the word y is sequence of words $x^0, x^1, \ldots, x^n \in \Sigma$ such that $x^0 = x, x^n = y$ and $\rho(\sigma(x^i), x^{i+1}) < \varepsilon$.
- The words $x, y \in \Sigma$ are in the *chain relation* $\mathcal{C}$ if for every $\varepsilon > 0$ there exists an ε-chain (of nonzero length) from x to y.

ε-chain in picture



ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$\begin{array}{rl} \dots aab & aa\dots \\ \dots aabaa\dots aa & aa\dots \\ & \text{Hop!} \\ \dots aa\dots a & a\dots aabaa\dots \\ \dots aa & baa\dots \end{array}$$

We have managed to shift the word to the right.

ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$\begin{array}{rl} \dots aab & aa\dots \\ \dots aabaa\dots aa & aa\dots \\ & \text{Hop!} \\ \dots aa\dots a & a\dots aabaa\dots \\ \dots aa & baa\dots \end{array}$$

We have managed to shift the word to the right.

ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$\begin{array}{rl} \dots aab & aa\dots \\ \dots aabaa\dots aa & aa\dots \\ & \text{Hop!} \\ \dots aa\dots a & a\dots aabaa\dots \\ \dots aa & baa\dots \end{array}$$

We have managed to shift the word to the right.

ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$\begin{array}{rl} \dots aab & aa\dots \\ \dots aabaa\dots aa & aa\dots \\ & \text{Hop!} \\ \dots aa\dots a & a\dots aabaa\dots \\ \dots aa & baa\dots \end{array}$$

We have managed to shift the word to the right.

ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$
\begin{array}{rl}
\dots aab & aa\dots \\
\dots aabaa\dots aa & aa\dots \\
 & \text{Hop!} \\
\dots aa\dots a & a\dots aabaa\dots \\
\dots aa & baa\dots
\end{array}
$$

We have managed to shift the word to the right.

ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$\begin{array}{rl} \dots aab & aa\dots \\ \dots aabaa\dots aa & aa\dots \\ & \text{Hop!} \\ \dots aa\dots a & a\dots aabaa\dots \\ \dots aa & baa\dots \end{array}$$

We have managed to shift the word to the right.

ε-chain and jumps

Take the subshift $\{a^{-\infty}ba^{\infty}, a^{\infty}\}$. Let the empty space denote the boundary between zeroth and first letter. Then we can produce for example this ε-chain:

$$
\begin{array}{rl}
\dots aab & aa\dots \\
\dots aabaa\dots aa & aa\dots \\
& \text{Hop!} \\
\dots aa\dots a & a\dots aabaa\dots \\
\dots aa & baa\dots
\end{array}
$$

We have managed to shift the word to the right.

- How to describe the chain relation in a general sofic subshift Σ?
- The main idea: We can jump between some vertices of G.
- Call such pairs of vertices *linked*.

- How to describe the chain relation in a general sofic subshift Σ?
- The main idea: We can jump between some vertices of G.
- Call such pairs of vertices *linked*.

- How to describe the chain relation in a general sofic subshift Σ?
- The main idea: We can jump between some vertices of G.
- Call such pairs of vertices *linked*.

Linked vertices

- Call two vertices u, v of a labelled graph G *linked* if for any length n there exists a word w of length n that has presentations beginning in both u, v and not leaving the components of u, v.
- By joining all pairs of linked vertices we obtain the *linking graph* $G/_\approx$.
- We have a natural projection of G onto $G/_\approx$.

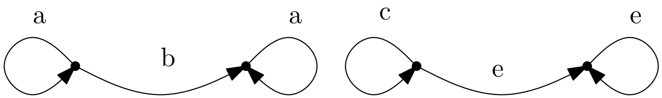
Linked vertices

- Call two vertices u, v of a labelled graph G *linked* if for any length n there exists a word w of length n that has presentations beginning in both u, v and not leaving the components of u, v.
- By joining all pairs of linked vertices we obtain the *linking graph* $G/_{\approx}$.
- We have a natural projection of G onto $G/_{\approx}$.

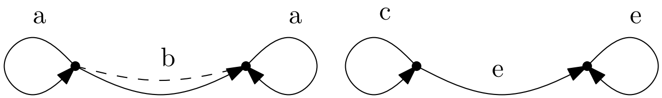
Linked vertices

- Call two vertices u, v of a labelled graph G *linked* if for any length n there exists a word w of length n that has presentations beginning in both u, v and not leaving the components of u, v.
- By joining all pairs of linked vertices we obtain the *linking graph* $G/_{\approx}$.
- We have a natural projection of G onto $G/_{\approx}$.

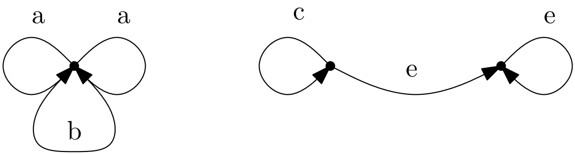
Linked vertices in picture



Linked vertices in picture



Linking graph in picture



Components of $G/_{\approx}$

- The components of the graph $G/_{\approx}$ can be partially ordered by the relation $K \leq L$ meaning “there exists a walk from K to L”.
- For x infinite word define $\alpha(x)$ and $\omega(x)$ as the components of $G/_{\approx}$ where the image of a presentation of x begins and ends.
- The components $\alpha(x), \omega(x)$ are well-defined: They always exist and do not depend on the choice of presentation of x.

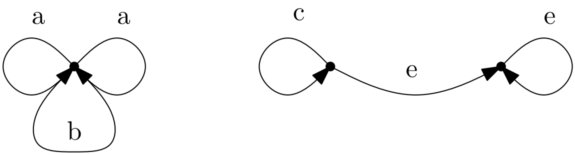
Components of $G/_{\approx}$

- The components of the graph $G/_{\approx}$ can be partially ordered by the relation $K \leq L$ meaning “there exists a walk from K to L”.
- For x infinite word define $\alpha(x)$ and $\omega(x)$ as the components of $G/_{\approx}$ where the image of a presentation of x begins and ends.
- The components $\alpha(x), \omega(x)$ are well-defined: They always exist and do not depend on the choice of presentation of x.

Components of $G/_{\approx}$

- The components of the graph $G/_{\approx}$ can be partially ordered by the relation $K \leq L$ meaning "there exists a walk from K to L".
- For x infinite word define $\alpha(x)$ and $\omega(x)$ as the components of $G/_{\approx}$ where the image of a presentation of x begins and ends.
- The components $\alpha(x), \omega(x)$ are well-defined: They always exist and do not depend on the choice of presentation of x.

The components of $G/_{\approx}$ in picture



K_1 is incomparable with K_2, K_3 and $K_2 \leq K_3$.

$$\alpha(a^{\infty}) = \omega(a^{\infty}) = K_1$$

The components of $G/_{\approx}$ in picture

K_1 is incomparable with K_2, K_3 and $K_2 \leq K_3$.

$$\alpha(a^\infty) = \omega(a^\infty) = K_1$$

The components of $G/_{\approx}$ in picture

K_1 is incomparable with K_2, K_3 and $K_2 \leq K_3$.

$\alpha(a^\infty) = \omega(a^\infty) = K_1$

The components of $G/_{\approx}$ in picture

K_1 is incomparable with K_2, K_3 and $K_2 \leq K_3$.

$$\alpha(a^\infty) = \omega(a^\infty) = K_1$$

Theorem about chain relation

Theorem

Let Σ be a sofic subshift, G its labelled graph. Let $x, y \in \Sigma$. Then $(x, y) \in \mathcal{C}$ iff $\omega(x) \leq \alpha(y)$ or $y = \sigma^n(x)$ for some $n > 0$.

Other properties of subshifts

- A subshift is *chain-transitive* if every two its words are in the chain relation.
- A susbift is *chain-mixing* if for every two words $x, y \in \Sigma$ and $\varepsilon > 0$ exists a k such that for all $n > k$ there exists an ε-chain from x to y of length n.
- Chain transitivity is often used when describing dynamic systems, the chain mixing property can be useful for finding attractors of celluar automata.

Other properties of subshifts

- A subshift is *chain-transitive* if every two its words are in the chain relation.
- A susbift is *chain-mixing* if for every two words $x, y \in \Sigma$ and $\varepsilon > 0$ exists a k such that for all $n > k$ there exists an ε-chain from x to y of length n.
- Chain transitivity is often used when describing dynamic systems, the chain mixing property can be useful for finding attractors of celluar automata.

Other properties of subshifts

- A subshift is *chain-transitive* if every two its words are in the chain relation.
- A susbift is *chain-mixing* if for every two words $x, y \in \Sigma$ and $\varepsilon > 0$ exists a k such that for all $n > k$ there exists an ε-chain from x to y of length n.
- Chain transitivity is often used when describing dynamic systems, the chain mixing property can be useful for finding attractors of celluar automata.

Characterising chain transitive and chain mixing subshifts

Theorem

Let Σ be a sofic subshift, G its essential graph. Then Σ is chain transitive iff $G/_{\approx}$ is (strongly) connected.

Theorem

Let Σ be a sofic subshift, G its essential graph. Then Σ is chain mixing iff $G/_{\approx}$ is (strongly) connected and aperiodic.

Characterising chain transitive and chain mixing subshifts

Theorem

Let Σ be a sofic subshift, G its essential graph. Then Σ is chain transitive iff $G/_{\approx}$ is (strongly) connected.

Theorem

Let Σ be a sofic subshift, G its essential graph. Then Σ is chain mixing iff $G/_{\approx}$ is (strongly) connected and aperiodic.

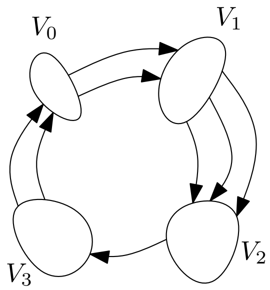
Intermezzo: Aperiodic graphs

A graph is *periodic* iff $V(G)$ can be partitioned into $n > 1$ disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ such that every edge $e \in E(G)$ leads from some $v \in V_k$ to some $u \in V_{k+1}$ for a suitable k.

A graph is *aperiodic* if it is not periodic.

Intermezzo: Aperiodic graphs

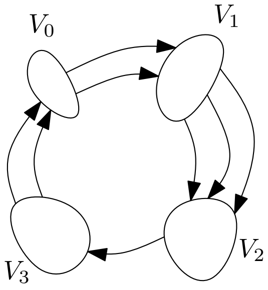
A graph is *periodic* iff $V(G)$ can be partitioned into $n > 1$ disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ such that every edge $e \in E(G)$ leads from some $v \in V_k$ to some $u \in V_{k+1}$ for a suitable k.



A graph is *aperiodic* if it is not periodic.

Intermezzo: Aperiodic graphs

A graph is *periodic* iff $V(G)$ can be partitioned into $n > 1$ disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ such that every edge $e \in E(G)$ leads from some $v \in V_k$ to some $u \in V_{k+1}$ for a suitable k.



A graph is *aperiodic* if it is not periodic.

Attractors

Informally, an attractor of a dynamic system is a closed set that attracts all trajectories from its neighbourhood.

Theorem

The attractors of the dynamic system (Σ, σ) are precisely all the subshifts described by the preimages of nonempty terminal subgraphs of $G/_{\approx}$.

Attractors

Informally, an attractor of a dynamic system is a closed set that attracts all trajectories from its neighbourhood.

Theorem

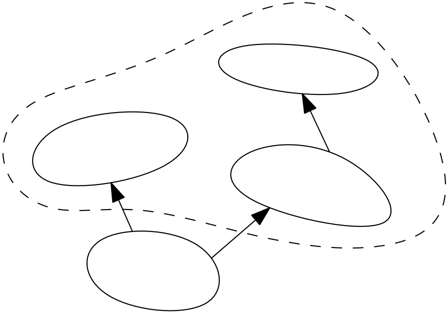
The attractors of the dynamic system (Σ, σ) *are precisely all the subshifts described by the preimages of nonempty terminal subgraphs of* $G/_{\approx}$.

Intermezzo: Terminal subgraphs

A subgraph H of $G/_{\approx}$ is terminal iff there are no edges leading from $V(H)$ to $V(G) \setminus V(H)$.

Intermezzo: Terminal subgraphs

A subgraph H of $G/_{\approx}$ is terminal iff there are no edges leading from $V(H)$ to $V(G) \setminus V(H)$.



Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
  - Chain transitivity
  - The chain-mixing property
  - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
  - Chain transitivity
  - The chain-mixing property
  - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
  - Chain transitivity
  - The chain-mixing property
  - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
  - Chain transitivity
  - The chain-mixing property
  - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
  - Chain transitivity
  - The chain-mixing property
  - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
  - Chain transitivity
  - The chain-mixing property
  - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

Summary

- Using the linking graph we can describe the chain relation in sofic subshifts.
- Using this knowledge we can explicitely describe:
    - Chain transitivity
    - The chain-mixing property
    - The attractors of the subshift dynamic system
- All three above properties can be checked algorithmically.

In the future, I plan to further investigate the properties of the linking graph and its connection to sofic subshifts.

The End

Thanks for your attention.

The End

Thanks for your attention.