# The interpretability lattice of clonoids is distributive

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# Clonoids (AKA minions AKA minor closed sets)

- A functional clonoid $\mathcal{C}$ on sets $A, B$ is a nonempty family of operations from $A$ to $B$ closed under taking minors
- Taking minors: $\sigma\colon [n] \to [m]$ sends $n$-ary $f$ to $m$-ary $f^\sigma$ where

$$f^\sigma(x_1, \dots, x_m) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

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# Clonoid homomorphisms

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- Another view: homomorphism sends identities true in $\mathcal{C}$ to identities of $\mathcal{D}$ – can interpret $\mathcal{C}$ in $\mathcal{D}$
- Example:

$$f(x, x, y) \approx g(x, y, y, z) \Rightarrow \phi(f)(x, x, y) \approx \phi(g)(x, y, y, z)$$

- For $\mathbb{A}, \mathbb{B}, \mathbb{A}', \mathbb{B}'$ finite relational structures $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to \mathrm{Pol}(\mathbb{A}', \mathbb{B}')$ gives a reduction from $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ to $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$ [Bulín, Krokhin, Opršal; 2018]

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# Interpretability lattice

- $\mathcal{C} \to \mathcal{D}$ is a quasiorder – factorize and look at the poset $\mathcal{L}$
- Libor Barto: A lot of categories embed (fully) into $\mathcal{L}$ (it is alg-universal)
- Warning: $\mathcal{L}$ is class-size
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- $\mathcal{C}$ goes from $A_1$ to $B_1$ and $\mathcal{D}$ goes from $A_2$ to $B_2$
- $\mathcal{C} \wedge \mathcal{D}$ has operations $(f, g)$ where $f$ and $g$ are operations in $\mathcal{C}$ and $\mathcal{D}$
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- Operations of $\mathcal{C} \vee \mathcal{D}$: Disjoint union of $\mathcal{C}$ and $\mathcal{D}$
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Thank you for your attention.