Name:

Convex optimization

Homework set 9

Date due: **December 11 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	2	
3	2	
4	2	
5	2	
Σ	10	

Problem 1. The problem

$$\begin{array}{rl} \text{minimize} & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

has the optimal solution $x_1^\star = x_2^\star = x_3^\star = 1/3$. Construct the Lagrangian of this problem and use the KKT conditions to calculate the dual optimal solution of this problem (that is, the optimal solution of the dual to (P)). Solve this problem by hand, not on a computer.

Problem 2. Find the dual problem to the following version of least squares (beware, the result will be different than in Problem 1 of series 8, even though the problems are equivalent; the difference is that we are minimizing the 2-norm rather than the square of the 2-norm):

$$\begin{array}{rl} \text{minimize} & \|\mathbf{y}\|_2 \\ \text{subject to} & \mathbf{y} = A\mathbf{x} - \mathbf{b} \end{array}$$

Problem 3. Let A be an $2 \times n$ matrix whose both rows are "random" in the sense that for all $0 < i < j < k \le n$ and all $u_i, u_j, u_k \in \{-1, 1\}$ the matrix

$$\begin{pmatrix} a_{1i} & a_{1j} & a_{1k} \\ a_{2i} & a_{2j} & a_{2k} \\ u_i & u_j & u_k \end{pmatrix}$$

has rank 3 (that is, its rows are all linearly independent). Prove that then the optimal solution $\mathbf{x}^\star$ of the problem

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{x}\|_1 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

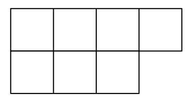
has at most two nonzero entries $x_i^\star \neq 0$.

Hint: Rewrite the problem into equivalent form

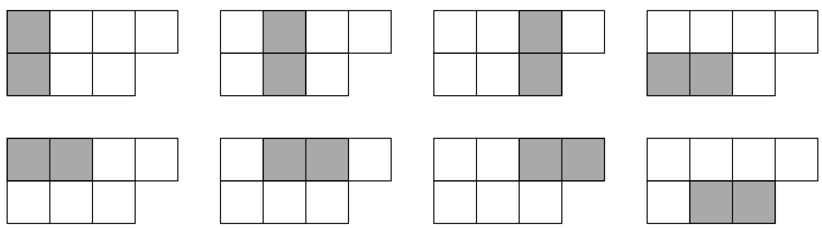
$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T\mathbf{x}^+ + \mathbf{1}^T\mathbf{x}^- \\ \text{subject to} \quad & A\mathbf{x}^+ - A\mathbf{x}^- = \mathbf{b} \\ & \mathbf{x}^+, \mathbf{x}^- \succeq 0 \end{aligned}$$

and examine the dual. Complementary slackness will help you here.

Problem 4. Consider the following version of the game of Battleship: There is just one round. Player 1 places a 1×2 ship somewhere in the irregular playing field which looks like in the picture



with 8 possible ship positions:



Player 2, not knowing Player 1's choice, picks one of the 7 squares to shoot at. If Player 2's shot hits the ship, Player 1 loses a point and Player 2 gains a point, otherwise Player 1 gains a point and Player 2 loses a point. Use CVXOPT/CVXOPT to calculate the worst-case optimal strategy for both players and the value of this game.

Explain your program and list the value of the game and optimal strategies here on paper and send your code to Jiří.

Hint: The default LP solvers calculate both primal and dual solutions of LPs, so it is enough to solve just one optimization problem.

Problem 5. In the lecture I told you that given a zero-sum game with the payoff matrix A and worst-case optimal strategies for P1 and P2 $\mathbf{p}^\star$, $\mathbf{q}^\star$ the value of the game is $(\mathbf{q}^\star)^T A\mathbf{p}^\star$ and that

$$\inf_{\substack{\mathbf{q}\\ \text{P2 strategy}}} \mathbf{q}^T A\mathbf{p}^\star = (\mathbf{q}^\star)^T A\mathbf{p}^\star = \sup_{\substack{\mathbf{p}\\ \text{P1 strategy}}} (\mathbf{q}^\star)^T A\mathbf{p}.$$

However, we did not really prove this – we will do that now. Let (P) be the problem "Find a worst case optimal strategy for Player 2" as in the lecture:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T\mathbf{q} \preceq t\mathbf{1} \\ & \mathbf{q} \succeq 0 \\ & \mathbf{1}^T\mathbf{q} = 1. \end{array}$$

Let $L(\mathbf{q}, t, \mathbf{p}, \lambda, \mu)$ be the Lagrangian of (P) (where the vector $\mathbf{p}$ consists of the dual variables corresponding to the inequality $A^T\mathbf{q} \preceq t\mathbf{1}$ and λ corresponds to $\mathbf{q} \succeq 0$).

a) Show that if $\mathbf{p}$ is a P1 strategy, $\mathbf{q}$ is a P2 strategy and complementary slackness holds for λ and $\mathbf{q}$, then $L(\mathbf{q}, t, \mathbf{p}, \lambda, \mu) = \mathbf{q}^T A\mathbf{p}$.

b) Use a) to prove that $(\mathbf{q}^\star)^T A\mathbf{p}^\star$ is the optimal value of (P) (and thus the value of the game).

c) Prove that

$$\inf_{\substack{\mathbf{q}\\ \text{P2 strategy}}} \mathbf{q}^T A\mathbf{p}^\star = (\mathbf{q}^\star)^T A\mathbf{p}^\star = \sup_{\substack{\mathbf{p}\\ \text{P1 strategy}}} (\mathbf{q}^\star)^T A\mathbf{p}.$$

   Beware of the fact that for $L(\mathbf{q}, t, \mathbf{p}, \lambda, \mu) = \mathbf{q}^T A\mathbf{p}$ you need extra assumptions on $\mathbf{p}, \mathbf{q}, \lambda$.

   Hint for b) and c): Use the properties of L that we discussed when talking about complementary slackness and the KKT conditions.

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.