Name:

Convex optimization

Homework set 7

Date due: **November 27 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	2	
3	2	
4	2	
5	2	
Σ	10	

Problem 1. Let $K \subset \mathbb{R}^n$ be a closed cone. Prove that K contains a line if and only if $K^\star$ does not contain n linearly independent vectors. You can use without proof the result that $(K^\star)^\star = \overline{K}$ ($\overline{K}$ is the closure of K; have not proved this yet).

Problem 2. Let $L \subset \mathbb{R}^n$ be a cone that contains n linearly independent vectors. Prove that L has nonempty interior. Use this result together with the claim from Problem 1 to prove that if K is proper then $K^\star$ has nonempty interior.

Hint 1: Take conic combinations of the linearly independent vectors.

Hint 2: Remember the triangle inequality $\|\sum_{i=1}^n x_i\|_2 \leq \sum_{i=1}^n \|x_i\|_2$.

Problem 3. Use Farkas' lemma (without a computer) to prove that the system of equalities and inequalities

$$\begin{aligned} 2x_3 - x_4 - x_5 &= 3 \\ x_1 + x_2 - x_3 + x_4 - x_6 &= -2 \\ x_1 + x_2 + x_3 + x_7 &= 0 \\ x_1, x_2, \ldots, x_7 &\geq 0 \end{aligned}$$

has no solution.

Problem 4. Prove that the interior of S^n_+ is S^n_{++}. Hint: Prove two inclusions and recall that for any $\mathbf{v} \in \mathbb{R}^n$ we have $\mathbf{v}\mathbf{v}^T \in S^n_+$.

Problem 5 (two-way partitioning)**.** Consider the problem of how to divide n people into two groups to minimize the total unhappiness. For i-th and j-th person let $W_{i,j}$ be the amount of unhappiness generated when i and j share a group. Assume that the matrix W is symmetric.

The division into two groups will be encoded by a vector $\mathbf{x} \in \mathbb{R}^n$ with components ± 1. The total unhappiness amount will be

$$\sum_{i,j=1}^{n} W_{ij} x_i x_j = \mathbf{x}^T W \mathbf{x}$$

Sadly, minimizing unhappiness in this situation is not a convex problem (it is an integer programming problem), but we can at least get a lower bound on the possible unhappiness.

1. Formulate the dual problem to

$$\begin{array}{ll} \text{minimize} & \mathbf{x}^T W \mathbf{x} \\ \text{subject to} & x_i^2 - 1 = 0 \quad \forall i = 1, \ldots, n \end{array}$$

   Find an equivalent form of this dual that is a semidefinite programming problem.

2. What does weak duality theorem tell us about optimal value of the original problem and $d^\star$?

3. Find $d^\star$ using CVXOPT/CVXPY for $n = 6$ with the matrix

$$W = \begin{pmatrix} 0 & 1 & -1 & 0 & 4 & -1 \\ 1 & 0 & 2 & 3 & -3 & 2 \\ -1 & 2 & 0 & 1 & -2 & 1 \\ 0 & 3 & 1 & 0 & 2 & 1 \\ 4 & -3 & -2 & 2 & 0 & -1 \\ -1 & 2 & 1 & 1 & -1 & 0 \end{pmatrix}.$$

   Send your code and the optimal solution of the dual to Jiří.

4. Try to find a solution with unhappiness close to $d^\star$ by hand. Discuss how close/far you got.

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.