Name:

Convex optimization

Homework set 5

Date due: **November 13 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	2	
3	2	
4	4	
Σ	10	

Problem 1 (Matrix trace and linear forms)**.** In semidefinite programming, we often talk about traces of products of matrices. Trace of a product turns out to be a scalar product on S^n. In this problem, your goal will be to verify a consequence of this fact.

Let $n \in \mathbb{N}$. Let f be a linear mapping $S^n \to \mathbb{R}$ (linear in the linear algebra sense, i.e. a linear form). Prove that then there exists a matrix $C \in S^n$ such that $f(X) = \mathrm{Tr}(CX)$ for all $X \in S^n$.

Problem 2. Let P be the SOCP problem

$$\begin{aligned} \text{minimize } & t \\ \text{subject to } & \|(x_1, 2x_2+2)\|_2 \le t, \\ & \|(1, x_2)\|_2 \le x_1 + 1. \end{aligned}$$

Construct a cone K, vectors $\mathbf{c}, \mathbf{g}$ and a matrix F so that (P) is equivalent to the cone program

$$\begin{aligned} \text{minimize } & \mathbf{c}^T \mathbf{y} \\ \text{subject to } & F\mathbf{y} + \mathbf{g} \preceq_K \mathbf{0}. \end{aligned}$$

Hint: K needs to somehow incorporate both second order cones in the constraints of P. A good place to search for K is $\mathbb{R}^6$.

Problem 3. We are going on the traditional Halloween zombie hunting expedition. We have a backpack that can take 10 kg of equipment and we want to maximize the sum of the utility of objects we take. In real life, we can either pack a object into the backpack or not, which is not a continuous situation (this is called *integer programming*). To make the problem convex, we will pretend that it is possible to pack e.g. 60 % of an axe and get 60 % of axe's utility; this is called a linear relaxation of the original problem.

Here is the table of objects we can take with us (the unit of utility is a destroyed zombie (dz)):

Object	Weight (kg)	Utility (dz)
Map	0.2	1
Torchlight	0.6	2
Katana	1.2	4
Guitar (out of tune)	2.55	6
Laptop	2.65	6
Firefighter's axe	3.15	7
1990s cell phone	3.2	7
Bowling ball	3.35	8
Halberd of zombie slaying	3.55	8
Angry cat	3.95	9
Skateboard	4.1	9
Collected works of V. Jarník	4.3	10
Bear trap	4.55	10
Chainsaw	9	20

Formulate the relaxed problem as an LP and find an optimal numerical solution. You do not have to submit your code, but do state your LP here and explain how it models the packing problem. Moreover, send Jiří the values of your optimal solution.

Compare the optimal value of the relaxed problem with the best value of the real life problem (where we either take an item with us or not) that you can find (by hand).

Note: It is not possible to pack more copies of a given object (e.g., packing 50 maps is not the solution).

Problem 4 (von Neumann growth problem, see p. 152)**.** We are planning activities in a math department in 2019 and 2020. We have 5 possible activities that we can do with different intensities $x_1, \dots, x_n$ in 2019 and we can do the same activities again in 2020 with intensities $x_1^+, \dots, x_n^+$. The big boss has decided that from 2019 to 2020 we should maximize the rate of growth across all activities, defined as $g(\mathbf{x}, \mathbf{x}^+) = \min_{i=1}^{5} x_i^+/x_i$.

Besides the activities, we have 5 resources (office supplies, coffee, money, fame and happines). For simplicity, the only rule about resources is that resources consumed in 2020 must be less than or equal to resources produced in 2019. Also, we will assume that we will be always doing a positive amount of all activities.

The activities are

1. Writing papers consumes x_1 units of office supplies, x_1 coffee and x_1 happiness and produces $3x_1$ of fame.
2. Napping consumes x_2 units of fame and produces x_2 units of happiness.
3. Applying for grants consumes $3x_3$ office supplies and $3x_3$ happiness and produces $10x_3$ money.
4. Drinking coffee consumes x_4 coffee and produces x_4 happiness and x_4 fame.
5. Administration consumes $2x_5$ money and $3x_5$ happiness and produces $5x_5$ office supplies and $2x_5$ coffee.

a) Formulate (on paper) the problem of maximizing the (relative) growth rate of least-growing activity as a generalized linear-fractional programming problem P. Note that since the problem is homogeneous, only the ratios between the x_i's matter; it might be a good idea to assume that $x_1 + x_2 + \cdots + x_5 = 1$.

b) Let $\alpha \in \mathbb{R}_{++}$ be fixed. Starting from P, design a linear program Q so that from the optimal value of Q we can deduce the answer to the question "Do there exist activity levels so that $g(\mathbf{x}, \mathbf{x}^+)$ is at least α?"

c) Using a computer, CVXOPT/CVXPY LP solvers, and some common sense upper and lower bounds on growth and binary search, find an activity plan $(\mathbf{x}, \mathbf{x}^+)$ that is 0.005-close to the optimum value of P. CVXPY allows you to directly solve quasiconvex problems; do not use this feature and instead write your own code. *Send* your code and your numerical solution to Jiří.

   Note: Depending on how you implement things, you might run into strange behavior of numerical solvers. If your code looks mathematically correct, but it outputs nonsense and you cannot figure out what is wrong, let us know.

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.