Name:

Convex optimization

Homework set 4

Date due: **November 6 2019, 17:21**

Explain your reasoning in all the problems.

Problem	Pts max	Pts
1	2	
2	2	
3	2	
4	4	
Σ	10	

Problem 1. Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = -57\log(3x_1+7x_2-10)+x_2^2-x_1x_2+5x_1^2+\text{max. eigenvalue of } \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1+x_2 \end{pmatrix}$$

is convex.

Problem 2. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *log-concave* if $f(x) > 0$ for all $x \in \operatorname{dom} f$ and $\log f(x)$ is concave. Prove that the function

$$f(x_1, x_2, x_3) = (1 - e^{-x_1})(1 - e^{-2x_2})(1 - e^{-5x_3})$$

with domain $\mathbb{R}^3_{++}$ is log-concave.

Problem 3. Prove by induction on n that the function

$$f_n(x_1, x_2, \ldots, x_n) = \ln(\exp(x_1) + \exp(x_2) + \cdots + \exp(x_n))$$

is convex for any $n \in \mathbb{N}$. You can use without proof that f_2 is convex (this was a problem in the previous homework set). This problem can also be solved by examining the Hessian of f_n, but we ask you not to do that (it is tedious, anyway).

Hint: Try to express f_n as a composition of several f_k's for $k < n$.

Problem 4. We are competing in rocket slalom on a 2D plane. At time t, our rocket has the x coordinate equal to t; we only control the rocket's velocity in the y direction by firing side thrusters.

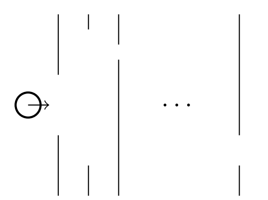
We will discretize time in 1/10 of a second increments, only considering the situation in $t = 0, 0.1, 0.2, \ldots, 9.9, 10$. The trajectory of our rocket is described by 101 triples (x_t, y_t, v_t) for $t \in \{0, 0.1, \ldots, 10\}$. At the beginning, we have $x_0 = y_0 = v_0 = 0$.

The motion equations for our rocket are

$$\begin{aligned} x_t &= t \\ y_{t+0.1} &= y_t + \frac{1}{10} v_t \\ v_{t+0.1} &= v_t + \Delta_t, \end{aligned}$$

where Δ_t is the impulse gained by burning fuel at time t – this is what we control. The impulse can be positive or negative ("up" or "down"). An impulse of Δ_t costs us Δ_t^2 fuel.

The slalom racetrack consists of a series of 10 barriers in the y direction at x positions 1, 2,..., 10 (see the sketch to the right). To complete the race without crashing, we need to make sure that for each $t = 1, \ldots, 10$ we have $y_t \in [a_t, b_t]$, where a_t, b_t are the y coordinates of the ends of the openings in the barriers. We know a_t, b_t in advance; they are constant.



a) State the problem "Given a_t, b_t's for $t \in \{1, \ldots, 10\}$, find a sequence of Δ_t's that allows us to complete the race while minimizing fuel usage" as a quadratic programming problem here on paper. Explain how exactly your model works.

b) Implement your model in CVXOPT/CVXPY. You do not have to send us the program, but you will need the program for the next step.

c) Optimizing purely for fuel consumption leads to dangerous trajectories. Find some values of a_i, b_i such that for all $i = 1, \ldots, 10$ we have $|b_i - a_i| \geq 1$, yet the trajectory given by minimizing fuel comes closer than 10^{-3} to one of the barriers for at least one t. Explain how you got your a_i, b_i's. Either write the values a_i, b_i and summarize the trajectory x_t here (if your solution is human-readable), or send the numbers to Jiří by e-mail.

d) Propose how to modify the optimization problem from a) to make the path of the rocket safer (it is OK to burn some extra fuel to do that). Discuss the drawbacks of your proposed change.

You can consult with your friends when solving the homework, but you have to **write** your solutions (including Python code) **on your own** and **do not show your fininished solutions** to your peers before the due date.