Convex optimization

Tutorial 7

Problem 1. Prove that the system of inequalities below has no solution

$$-x_1 - x_2 \geq -1$$
$$x_1 \geq 1$$
$$x_2 \geq 1.$$

How is this related to duality for LPs?

Problem 2. State the dual problem to the LP

$$\begin{aligned} \text{minimize } & x_1 - x_2 + 2x_3 \\ \text{subject to } & x_1 + x_2 + x_3 - 10 \leq 0 \\ & -x_1 \leq 0 \\ & x_2 - x_3 = 1. \end{aligned}$$

Problem 3. Let (P) be a problem without (explicitly given) constraints "minimize $f(A\mathbf{x} + \mathbf{b})$". The function f is convex, and defined on a convex (yet compicated) set $\mathcal{D}$, A is a matrix, and b a vector. How does a dual problem to (P) look like? How does a dual problem to (P') look like, where (P') is defined as follows:

$$\begin{aligned} \text{minimize } & f(\mathbf{y}) \\ \text{subject to } & \mathbf{y} = A\mathbf{x} + \mathbf{b} \end{aligned}$$

Which dual problem looks more useful?

Problem 4. Let $n \in \mathbb{N}$. Let P be the (minimization version of) the entropy maximization problem

$$\begin{aligned} \text{minimize } & \sum_{i=1}^{n} x_i \ln x_i \\ \text{subject to } & a_1 x_1 + \cdots + a_n x_n = b \\ & x_1 + x_2 + \cdots + x_n = 1. \end{aligned}$$

State the dual problem to P and think about how to solve it analytically.

Problem 5 (complementarity)**.** Prove, that if for all $i = 1, \ldots, m$ is $\lambda_i \geq 0$, $f_i(\mathbf{x}^\star) \leq 0$ and

$$\sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}^\star) = 0,$$

then $\lambda_i > 0$ implies $f_i(\mathbf{x}^\star) = 0$, and $f_i(\mathbf{x}^\star) < 0$ implies $\lambda_i = 0$. Does $f_i(\mathbf{x}^\star) = 0$ imply $\lambda_i > 0$?