Tutorial 3

Problem 1. Is a function $f(x, y) = xy$ convex? Is it concave?

Problem 2. A function $\mathbb{R}^n \to \mathbb{R}$ is a *norm* if:

1. $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with equality if and only if $\mathbf{x} = \mathbf{0}$,
2. for all $t \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$ we have $\|t\mathbf{x}\| = |t|\|\mathbf{x}\|$, and
3. for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Prove that any norm is a convex function.

Problem 3. Let A be a regular $n \times n$ matrix, $\mathbf{x}_c \in \mathbb{R}^n$ a fixed vector. Prove that the ellipsoid $E = \{\mathbf{x}_c + A\mathbf{u} \colon \mathbf{u} \in \mathbb{R}^n,\ \|\mathbf{u}\|_2 \leq 1\}$ is a convex set.

Hint: Build E from a simpler convex set. The convexity of norms might also help.

Problem 4. Let $f(\mathbf{x}) = \max\{\mathbf{c}_1^T\mathbf{x} + b_1, \ldots, \mathbf{c}_k^T\mathbf{x} + b_k\}$ be a convex, piece–wise affine function, and A such a matrix, that $A\mathbf{x}$ makes sense.

Think, how to rewrite the problem 'minimize $f(\mathbf{x})$ under conditions $A\mathbf{x} \preceq 0$' as a linear programming problem.

Problem 5. Let f be a convex function. Prove that:

1. If f attains a local minimum at $\mathbf{x}$, then f in fact attains its global minimum at $\mathbf{x}$.
2. Prove that the set $\operatorname{argmin}(f) = \{\mathbf{x} \colon f(\mathbf{x}) \text{ is minimal}\}$ is convex.

Problem 6. Let f_1, f_2 be convex functions $\mathbb{R}^n \to \mathbb{R}$. Decide, for which values of $\lambda_1, \lambda_2 \in \mathbb{R}$ is the function

$$g(\mathbf{x}) = \lambda_1 f_1(\mathbf{x}) + \lambda_2 f_2(\mathbf{x})$$

always convex.

Problem 7. We own an ice cream factory and we are producing ice cream for one year from January to December. We have a good idea how much ice cream we can sell throughout the year: for the i-th month of the year we write z_i the amount (in tonnes) of ice cream we are able to sell during the month.

However, the demand for ice cream changes during the year. We can boost or slow the production, or we can store the ice cream. But all of these cost money: the change of production amount between months costs c per ton, the storage of ice cream costs s per month and ton.

Formulate a (linear) optimization problem describing how to cover the demand throughout the year while minimizing the cost – we start and finish with empty storage, the initial setting of production in January is free.