Tutorial 2

Problem 1. Is the set $\{\mathbf{x} \in \mathbb{R}^3 \colon x_1^2 + x_2^2 < x_3^2\}$ a cone? If so, is it a proper cone?

Problem 2. Let $\mathbf{a} \neq \mathbf{0}$ be an n-dimensional vector and $b \in \mathbb{R}$. Prove that the half-space $\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{a}^T\mathbf{x} \leq b\}$ is convex.

Problem 3. Prove that if $X \subset \mathbb{R}^n$ is closed under convex combinations of pairs of points, then X is closed under general convex combinations.

Problem 4. In the gas problem from last time, we needed to express a piecewise linear function in a linear program. We will try it again today. Figure out how to rewrite the following program as a linear programming program by adding one new variable and some constraints:

$$\begin{aligned} \text{minimize } & \max\{0, 3x - y\} \\ \text{subject to } & x, y \leq 3 \\ & x, y \geq 0 \end{aligned}$$

Problem 5. A function $\mathbb{R}^n \to \mathbb{R}$ is a *norm* if:

1. $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with equality if and only if $\mathbf{x} = \mathbf{0}$,
2. for all $t \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$ we have $\|t\mathbf{x}\| = |t|\|\mathbf{x}\|$, and
3. for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Prove that any norm is a convex function.

Problem 6. Let $K \subset \mathbb{R}^n$ be a proper cone. Show that the generalized inequality $\preceq_K$ satisfies for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and each $t \geq 0$

1. $\mathbf{x} \preceq_K \mathbf{x}$,
2. $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{z}$ implies $\mathbf{x} \preceq_K \mathbf{z}$,
3. if $\mathbf{x} \preceq_K y$ then $t\mathbf{x} \preceq_K t\mathbf{y}$, and
4. if $\mathbf{x}, \mathbf{y} \preceq_K \mathbf{0}$ then $\mathbf{x} + \mathbf{y} \preceq \mathbf{0}$.

Hint: If you are confused, try it for $K = \mathbb{R}^n_+$ first.

Bonus: Show that there is no $\mathbf{x}$ such that $\mathbf{x} \prec_K \mathbf{x}$.

Problem 7. Let $X \subset \mathbb{R}^n$ be a (convex) cone that is closed. Prove that if X contains the line $\{t\mathbf{a} + \mathbf{c} \colon t \in \mathbb{R}\}$, then X contains the line $\{t\mathbf{a} \colon t \in \mathbb{R}\}$.