Konvexní optimalizace

Tutorial 13

Problem 1. Let T be

a) an equilateral triangle

b) a rectangle

in $\mathbb{R}^2$. How does the Löwner-John elipsoid pro T look like?

Problem 2 (Exact line search by halving intervals)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function that is defined and differentiable on $[0, 1]$. Let $p^\star = \inf\{f(x)\colon x \in [0, 1]\}$. Prove that the following algorithm always returns a point x such that $f(x) - p^\star \leq \epsilon$:

> **Data:** f, $\epsilon > 0$, L such that for any $x \in [0, 1]$ we have $|f'(x)| < L$
> **Result:** $x \in [0, 1]$
> **if** $f'(0) \geq 0$ **then return** 0;
> **if** $f'(1) \leq 0$ **then return** 1;
> $l := 0, u := 1, x := 1/2$;
> **while** $u - l > \epsilon/L$ **do**
>   **if** $f'(x) > 0$ **then** $u := x$;
>   **else** $l := x$;
> $x = (l + u)/2$;
> **end**
> **return** x

Note: After you solve the problem, it might help to look at it as a problem of solving $f'(x) = 0$.

Problem 3 (BLS works)**.** Let us have some $\alpha \in (0, 1/2)$ and $\beta \in (0, 1)$. Suppose f is a convex, differentiable function with open domain and $\mathbf{x} \in \operatorname{dom} f$, $\mathbf{\Delta x} \in \mathbb{R}^n$ are such that $\nabla f(\mathbf{x})^T \mathbf{\Delta x} < 0$. Prove that then the backtracking line search with input $f, \mathbf{x}, \mathbf{\Delta x}, \alpha, \beta$ will

a) terminate, and

b) return a t such that $f(\mathbf{x} + t\mathbf{\Delta x}) < f(\mathbf{x})$.

A descent method is *affine invariant* if when the sequence of points for a function f and a starting point $\mathbf{x}^{(0)}$ is $\mathbf{x}^{(k)}$ and $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijective affine mapping, then the method for function $f \circ h^{-1}$ and the starting point $h(\mathbf{x}^{(0)})$ produces the sequence of points $h(\mathbf{x}^{(k)})$.

Problem 4. Prove that gradient descent is not affine invariant.

Problem 5. Prove that the Newton's method (with exact line search, say) is affine invariant.