Tutorial 10

Problem 1. Solve the least squares problem from the quiz in the situation when we know that the sequence of q's is $q_1 = 1$, $q_2 = 3$, $q_3 = 5$, and moreover $r_1 = 1, r_2 = 2$ and r_3 is a random variable with mean 3 and variance 1.

Problem 2 (Linearized golf)**.** We are trying to get our golf ball as close to the hole as possible. We start at point $(0, 0)$ and the hole is at $(10, 3)$. We are so good at golfing that we can choose x_1 and x_2 and if there was no wind the golf ball would land exactly at the point (x_1, x_2). However, a wind of unknown strength u is blowing and for any choice of x_1 and x_2 the ball will really land at $(x_1, x_2 + ux_1)$. We do not know u, only that the expected value of u is 0 and its variance is 2.

Formulate and solve the problem of selecting x_1, x_2 so that the expectation of the distance of our ball's landing point to the hole is as small as possible.

Problem 3. Consider the regularized approximation problem with the objective function (to be minimized) $\|A\mathbf{x} - \mathbf{b}\| + \gamma\|\mathbf{x}\|$. We can choose $\gamma = 0.01$ or $\gamma = 10$.

1. In which case are we going to get $A\mathbf{x}$ closer to $\mathbf{b}$?
2. In which case are we going to get a smaller $\mathbf{x}$?

Problem 4. Prove in detail that whenever $g\colon \mathbb{R} \to \mathbb{R}$ is a convex function with domain $\mathbb{R}$ such that $g(u) = u^2$ for $u \in [-1, 1]$ then for all $u \in \mathbb{R}$ we have $g(u) \geq \phi(u)$ where ϕ is the Huber penalty function for $M = 1$.

Problem 5. Suppose you have received a signal $\mathbf{b} \in \mathbb{R}^m$ that corresponds to a sound recording $\mathbf{x} \in \mathbb{R}^n$ via $A\mathbf{x} = b + \text{noise}$. You want to recover $\mathbf{x}$ and you also know that $\mathbf{x}$ should be smooth in the sense that x_i and x_{i+1} usually do not differ by much (no sudden huge jumps). Formulate a two-criterion problem that looks for an $\mathbf{x}$ that is both smooth and satisfies $A\mathbf{x} \approx b$.

Problem 6. Consider the Tichonov regularization problem "minimize $\|A\mathbf{x} - \mathbf{b}\|_2^2 + \gamma\|\mathbf{x}\|_2^2$" with the parameter $\gamma \geq 0$.

1. Find an explicit formula for the optimal solution $\mathbf{x}^\star$ when $\gamma > 0$.
2. Show that if $\gamma \to \infty$ then the optimal solution will tend to $\mathbf{x}^\star = \mathbf{0}$.
3. We let $\gamma \to 0$. Assuming that A^TA is regular, what is the limit of $\mathbf{x}^\star$?
4. We let $\gamma \to 0$ again, but this time A^TA need not be regular. Show that the limit $\mathbf{x}^\star$ exists and that it is the $\mathbf{x}$ with minimal 2-norm among all the vectors $\mathbf{x}$ minimizing $\|A\mathbf{x} - \mathbf{b}\|_2^2$.