CONTINUITY OF SOLUTIONS OF PARABOLIC AND ELLIPTIC EQUATIONS.*

By J. NASH.

Introduction. Successful treatment of non-linear partial differential equations generally depends on "a priori" estimates controlling the behavior of solutions. These estimates are themselves theorems about linear equations with variable coefficients, and they can give a certain compactness to the class of possible solutions. Some such compactness is necessary for iterative or fixed-point techniques, such as the Schauder-Leray methods. Alternatively, the a priori estimates may establish continuity or smoothness of generalized solutions. The strongest estimates give quantitative information on the continuity of solutions without making quantitative assumptions about the continuity of the coefficients.

The theory of non-linear elliptic equations in two independent variables is fairly well developed. (See [1] for a survey and bibliography.) An essential part is the a priori Hölder continuity estimate for solutions of uniformly elliptic equations, first proved by Morrey in 1938. All methods used to obtain this estimate have been quite special to two dimensions, utilizing, for example, complex analysis and quasi-conformal mappings (see [2]). The restriction to two variables has been due to this use of such special methods; except for the crucial a priori estimate, the theory is extensible (and in large part has been extended) to n dimensions and to parabolic equations. Our results fill this gap, and it should now be possible to build a general theory of non-linear parabolic and elliptic equations, free of dimension restrictions. Strictly speaking, our work needs some generalization to cover equations with lower order terms, systems, etc. This generalization can probably be accomplished fairly quickly.

In this paper, we consider linear parabolic equations of the form

$$\sum_{i,j} \partial[C_{ij}(x_1, x_2, \cdots, x_n, t)\partial T/\partial x_j]/\partial x_i = \partial T/\partial t, \text{ or}$$

$$\nabla \cdot (C(x,t) \cdot \nabla T) = T_t, \tag{1}$$

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where the C_{ij} form a symmetric real matrix $C(x,t)$ for each point x and time t. We assume there are universal bounds $c_2 \geqq c_1 > 0$ on the eigenvalues of C so that any eigenvalue θ_ν satisfies $c_1 \leqq \theta_\nu \leqq c_2$. This is the standard "uniform ellipticity" assumption. The continuity estimate for a solution $T(x,t)$ of (1) satisfying $|T| \leqq B$ and defined for $t \geqq t_0$ is

$$(2) \quad |T(x_1,t_1) - T(x_2,t_2)| \leqq BA\{[|x_1 - x_2|/(t_1-t_0)^{\frac{1}{2}}]^\alpha + [(t_2-t_1)/(t_1-t_0)]^{\frac{1}{2}\alpha/(1+\alpha)}\},$$

where $t_2 \geqq t_1 > t_0$. Here A and α are a priori constants which depend only on c_1 and c_2 and the space dimension n. As a corollary of our results on parabolic equations, we obtain a continuity estimate for solutions of elliptic equations. If $T(x)$ satisfies $\nabla \cdot (C(x) \cdot \nabla T) = 0$ in a region R and the same bounds c_1 and c_2 limit the eigenvalues of $C(x)$, then

$$(3) \qquad |T(x_1) - T(x_2)| \leqq BA'(|x_1 - x_2|/d(x_1,x_2))^{\alpha/(1+\alpha)},$$

where α is the α of (2) and A' is an a priori constant $A'(n, c_1, c_2)$, and where $|T| \leqq B$ in R and $d(x_1,x_2)$ is the lesser of the distances of the points x_1 and x_2 from the boundary of R.

Our paper is arranged in six parts, each concluding with the attainment of a result significant in itself. Detailed proofs are given and all the results presented in [14] are covered. An appendix states further results, including continuity at the boundary in the Dirichlet problem, a Harnack inequality, and other results, stated without detailed proofs.

General remarks. The open problems in the area of non-linear partial differential equations are very relevant to applied mathematics and science as a whole, perhaps more so than the open problems in any other area of mathematics, and this field seems poised for rapid development. It seems clear, however, that fresh methods must be employed. We hope this paper contributes significantly in this way and also that the new methods used in our previous paper, reference [10], will be of value.

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid. These are a non-linear parabolic system of equations. Also the relationship between this continuum description of a fluid (gas) and the more physically valid statistical mechanical description is not well understood. (See [11], [12], and [13]). An interest in these questions led us to undertake this work. It became clear that nothing could be done about the continuum description of general fluid flow without the ability to handle

non-linear parabolic equations and that this in turn required an a priori estimate of continuity, such as (2).

Probably one should first try to prove a conditional existence and uniqueness theorem for the flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain *gross* types of singularity, such as infinities of temperature or density. (A gross singularity could arise, for example, from a converging spherical shock wave.) A result of this kind would clarify the turbulence problem.

The methods used here were inspired by physical intuition, but the ritual of mathematical exposition tends to hide this natural basis. For parabolic equations, diffusion, Brownian movement, and flow of heat or electrical charges all provide helpful interpretations. Moreover, to us, parabolic equations seem more natural than elliptic ones. It is certainly true in principle that the theory of parabolic equations includes elliptic equations as a specialization, and in applications an elliptic equation typically arises as the description of the steady state of a system which in general is described by a parabolic equation.

In our work, no difference at all appears between dimensions two and three. Only in one dimension would the situation simplify. The key result seems to be the moment bound (13); it opens the door to the other results. We had to work hard to get (13), then the rest followed quickly.

We are indebted to several persons and institutions in connection with this work, including Bers, Beurling, Browder, Carleson, Lax, Levinson, Morrey, Newman, Nirenberg, Stein and Wiener, the Alfred P. Sloan Foundation, the Institute for Advanced Study, M. I. T., N. Y. U., and the Office of Naval Research.

Part I: The Moment Bound.

More than enough is known about linear parabolic equations with variable coefficients to assure the existence of well behaved solutions for equations of the form (1) if we make strong (qualitative) restrictions on the C_{ij} and restrict the class of solutions to be considered. (See [3] through [7].) Therefore we assume: (a) The $C_{ij}(x, t)$ are uniformly C^∞, (b) $C_{ij}(x, t) = \sqrt{c_1 c_2}\,\delta_{ij}$ (Kronecker delta) for $|x| \geqq r_0$, some large constant. We consider only solutions $T(x, t)$ bounded in x for each t for which the solution is defined, i. e., $\max_x |T(x, t)|$ is finite.

Under these restrictions, any bounded measurable function $T(x, t_0)$ of x given at an initial time t_0 determines a unique continuation $T(x, t)$ defined for all $t \geqq t_0$ and C^∞ for $t > t_0$. Moreover, $T(x, t) \to T(x, t_0)$ almost everywhere as $t \to t_0$, and $\max_x |T(x, t)|$ is non-increasing in t. It is also known that fundamental solutions, which we discuss below, exist and have the general properties we state. (See [4], [7].)

After the a priori results are established, a passage to the limit can remove the restrictions on the C_{ij}. This is a standard device in the use of a priori estimates. The Hölder continuity (2) makes the family of solutions equicontinuous and forces a continuous limit (generalized) solution to exist. Furthermore, the maximum principle remains valid and with it the unique continuability of solutions bounded in space. The final result requires only measurability for the C_{ij}, plus the uniform ellipticity condition; and the a priori estimates then hold for the generalized solutions.

The use of fundamental solutions is very helpful with equations of the form (1). Our work is built around step by step control of the properties of fundamental solutions and most of the results concern them directly. A fundamental solution $T(x, t)$ has a "source point" x_0 and "starting time" t_0 and is defined and positive for $t > t_0$. Also, $\int T(x, t)\, dx = 1$ for every $t > t_0$, where dx is the volume element in n-space. As $t \to t_0$, the fundamental solution concentrates around ite source point; $\lim T(x, t)$ is zero unless $x = x_0$, in which case it is $+\infty$. Physically, a fundamental solution represents the concentration of a diffusant spreading from an initial concentration of unit weight at x_0 at time t_0.

All fundamental solutions are conveniently unified in a "characterizing function" $S(x, t, \bar{x}, \bar{t})$. For fixed $\bar{x}$ and $\bar{t}$ and as a function of x and t, S is a fundamental solution of (1) with source point $\bar{x}$ and starting time $\bar{t}$. Dually, for fixed x and t, S is a fundamental solution of the adjoint equation: $\nabla_{\bar{x}} \cdot [C(\bar{x}, t) \cdot \nabla_{\bar{x}} S] = -\partial S / \partial \bar{t}$, where time runs backwards. This duality enables us to use estimates for fundamental solutions in two ways on S.

The dependence of a bounded solution $T(x, t)$ on bounded initial data $T(x, t_0)$ is expressible through S:

$$T(x, t) = \int S(x, t, \bar{x}, t_0)\, T(\bar{x}, t_0)\, d\bar{x}; \tag{4}$$

in particular,

$$S(x_2, t_2, x_0, t_0) = \int S(x_2, t_2, x_1, t_1)\, S(x_1, t_1, x_0, t_0)\, dx_1. \tag{5}$$

These are standard relations. (5) reveals a reproductive property of fundamental solutions.

Now consider a special fundamental solution $T = T(x, t) = S(x, t, 0, 0)$ with source at the origin and starting time zero. Let

$$E = \int T^2\, dx,$$

then

$$E_t = 2\int T T_t\, dx = 2\int T \nabla \cdot (C \cdot \nabla T)\, dx = -2\int \nabla T \cdot C \cdot \nabla T\, dx,$$

by integration by parts. For any vector V, we have $c_1 | V |^2 \leqq V \cdot C \cdot V \leqq c_2 | V |^2$; therefore

$$-E_t \geqq 2c_1 \int | \nabla T |^2\, dx. \tag{6}$$

With (6) and a lower bound for $\int | \nabla T |^2\, dx$ in terms of E, we shall be able to bound E above, obtaining our first a priori estimate. To bound $\int | \nabla T |^2\, dx$ we employ a general inequality valid for any function $u(x)$ in n-space. For our purposes, we assume u is smooth and well behaved at infinity. E. M. Stein gave us the quick proof which follows below.

The Fourier transform of $u(x)$ is

$$v(y) = (2\pi)^{-n/2} \int e^{ix \cdot y} u(x)\, dx.$$

This has the familiar property

$$\int | v |^2 dy = \int | u |^2 dx.$$

The transform of $\partial u/\partial x_k$ is $iy_k v$; hence

$$\int | \partial u/\partial x_k |^2\, dx = \int y_k^2 | v |^2\, dy,$$

and

$$\int | \nabla u |^2\, dx = \sum_i \int (\partial u/\partial x_k)^2\, dx = \int | y |^2 | v |^2\, dy.$$

Finally,

$$| v | \leqq (2\pi)^{-n/2} \int | e^{ix \cdot y} | \cdot | u |\, dx = (2\pi)^{-n/2} \int | u |\, dx;$$

therefore, for any $\rho > 0$, we have

$$\int_{|y| \leqq \rho} | v |^2\, dy \leqq (\pi^{n/2} \rho^n / (n/2)\,!) \{ (2\pi)^{-n/2} \int | u |\, dx \}^2, \tag{a}$$

using the formula for the volume of an n-sphere. On the other hand,

$$\int_{|y| > \rho} | v |^2\, dy \leqq \int_{|y| > \rho} | y/\rho |^2 | v |^2\, dy = \rho^{-2} \int | \nabla u |^2\, dx. \tag{b}$$

If we choose the value of ρ minimizing the sum of the two bounds (a) and (b), we obtain a bound on $\int | v |^2\, dy = \int | u |^2\, dx$ in terms of $\int | u |\, dx$ and $\int | \nabla u |^2\, dx$. Solved for $\int | \nabla u |^2\, dx$, this is

$$\int |\nabla u|^2\,dx$$
$$\geqq (4\pi n/(n+2))[(n/2)!/(1+n/2)]^{2/n}[\int |u|\,dx]^{-4/n}[\int |u|^2\,dx]^{1+2/n}.$$

Applying the above inequality with $u = T$, remembering that $\int T dx = 1$, we obtain from (6)

$$-E_t \geqq kE^{1+2/n}.$$

This is the first use of a convention we now establish that k is a generic symbol for a priori constants which depend only on n, c_1, and c_2. Any two instances of k should be presumed to be different constants. Thus, from the above inequality, $(E^{-2/n})_t \geqq k$; hence $E^{-2/n} \geqq kt$ and

$$E \leqq kt^{-n/2}. \tag{7}$$

We used above the qualitative fact $\lim_{t\to 0} E = \infty$.

From this first bound (7) and the identity (5), we obtain

$$T(x,t) = \int S(x,t,\bar{x},t/2)\,S(\bar{x},t/2,0,0)\,d\bar{x},$$

whence

$$(T(x,t))^2 \leqq \int [S(x,t,\bar{x},t/2)]^2\,d\bar{x}\cdot\int [S(\bar{x},t/2,0,0)]^2\,d\bar{x} \leqq [k(t/2)^{-n/2}]^2.$$

Therefore

$$T \leqq kt^{-n/2}, \tag{8}$$

which is a pointwise bound, stronger than (7).

The key estimate controls the "moment" of a fundamental solution

$$M = \int rT\,dx = \int |x|\,T\,dx.$$

To prove $M \leqq kt^{\frac{1}{2}}$ is our first major goal. This is dimensionally the only possible form for a bound on M. The moment bound is essential to all subsequent parts of this paper.

We also define an "entropy."

$$Q = -\int T\log T\,dx. \tag{9}$$

From (8),

$$Q \geqq \int \min_x[-\log T]\,(T\,dx) \geqq -\log(kt^{-n/2})\int T\,dx,$$

hence

$$Q \geqq \pm k + \tfrac{1}{2}n\log t \tag{10}$$

because $\int T\,dx = 1$. The sharp result $Q \geqq \frac{1}{2}n\log(4\pi e c_1 t)$ is obtainable from a more sophisticated argument.

Our derivation of a bound on M requires a lower bound on M in terms of Q as a lemma. This inequality, which is $M \geqq ke^{Q/n}$, depends only on the facts $T \geqq 0$, $\int T\,dx = 1$. First observe that for any fixed λ,

$$\min_T (T \log T + \lambda T) = -e^{-\lambda-1}.$$

Let $\lambda = ar + b$, where $r = |x|$ and a and b are any constants, and integrate over space, obtaining

$$\int [T \log T + (ar + b)T]\,dx \geqq -e^{-b-1} \int e^{-ar}\,dx,$$

or

$$-Q + aM + b \geqq -e^{-b-1} a^{-n} D_n,$$

where D_n is the well known constant $2^n \pi^{\frac{1}{2}(n-1)} [\frac{1}{2}(n-1)]!$ related to the gamma-function and the surface of the $(n-1)$-sphere. Now set $a = n/M$ and $e^{-b} = (e/D_n) \cdot a^n$. Then $-Q + n + b \geqq -1$ or $n + 1 \geqq Q + \log(n/D_n) + \log(n/M)$; thus $n \log M + n \geqq Q + n \log n - \log D_n$, finally,

$$M \geqq (n/e\, D_n^{1/n})\, e^{Q/n} = ke^{Q/n}. \tag{11}$$

This ingenious proof, due to L. Carleson, gives an optimal constant.

The next inequality is a "dynamic" one, connecting the rates of change with time of M and Q. Differentiating (9),

$$\begin{aligned} Q_t = -\int (1 + \log T) T_t dx &= -\int (1 + \log T) \nabla \cdot (C \cdot \nabla T)\,dx \\ &= \int \nabla (\log T) \cdot C \cdot \nabla T\,dx \end{aligned}$$

after integration by parts. This can be rewritten

$$Q_t = \int \nabla (\log T) \cdot C \cdot \nabla (\log T)\,(T\,dx).$$

Since in general $V \cdot c_2 C \cdot V \geqq V \cdot C^2 \cdot V = |C \cdot V|^2$, where V is a vector, we have

$$\begin{aligned} c_2 Q_t \geqq \int |C \cdot \nabla (\log T)|^2 (T\,dx) &\geqq [\int |C \cdot \nabla \log T|\,(T\,dx)]^2 \\ &\geqq [\int |C \cdot \nabla T|\,dx]^2. \end{aligned}$$

Here we used the Schwarz inequality in the form $\int_0^1 f^2\,du \geqq [\int_0^1 f\,du]^2$ with du corresponding to $T\,dx$.

By analogous manipulations,

$$M_t = -\int \nabla r \cdot C \cdot \nabla T\,dx \quad \text{and} \quad |M_t| \leqq \int |\nabla r|\,|C \cdot \nabla T|\,dx,$$

hence,

$$|M_t| \leqq \int |C \cdot \nabla T|\,dx.$$

Combining inequalities,

$$c_2 Q_t \geqq (M_t)^2. \tag{12}$$

This is a powerful inequality. Q is defined as it is in order to obtain (12).

The three inequalities

$$Q \geqq \pm k + \tfrac{1}{2} n \log t \tag{10}$$

$$M \geqq k e^{Q/n} \tag{11}$$

$$c_2 Q_t \geqq (M_t)^2 \tag{12}$$

and the qualitative fact $\lim M = 0$, as $t \to 0$, suffice by themselves to bound above and below both M and Q, as functions of time. No further reference to the differential equation is needed.

From $M(0) = 0$ and (12),

$$M \leqq \int_0^t (c_2 Q_t)^{\frac{1}{2}}\, dt,$$

whence

$$k e^{Q/n} \leqq M \leqq \int_0^t (c_2 Q_t)^{\frac{1}{2}}\, dt.$$

Now define $nR = Q \mp k - \frac{1}{2} n \log t$ in such a way that $R \geqq 0$ corresponds to (10). Then $Q_t = nR_t + n/2t$, and we obtain

$$k t^{\frac{1}{2}} e^R \leqq M \leqq (n c_2)^{\frac{1}{2}} \int_0^t (1/2t + R_t)^{\frac{1}{2}}\, dt.$$

When a and $a + b$ are positive $(a+b)^{\frac{1}{2}} \leqq a^{\frac{1}{2}} + b/2a^{\frac{1}{2}}$, hence

$$\int_0^t (1/2t + R_t)^{\frac{1}{2}}\, dt \leqq \int_0^t (1/2t)^{\frac{1}{2}} dt + \int_0^t (t/2)^{\frac{1}{2}} R_t\, dt$$

$$\leqq (2t)^{\frac{1}{2}} + R(t/2)^{\frac{1}{2}} - \int_0^t R/(8t)^{\frac{1}{2}} dt \leqq (2t)^{\frac{1}{2}} + R(t/2)^{\frac{1}{2}}.$$

Here we used integration by parts and $R \geqq 0$ in the second and third steps. Applying this result,

$$k t^{\frac{1}{2}} e^R \leqq kM \leqq (2t)^{\frac{1}{2}} + R(t/2)^{\frac{1}{2}},$$

or

$$k e^R \leqq kM/t^{\frac{1}{2}} \leqq 2^{\frac{1}{2}}(1 + \tfrac{1}{2} R).$$

Clearly $k e^R$ increases faster in R than $2^{\frac{1}{2}}(1 + \frac{1}{2} R)$ so that R must be bounded above. Therefore $M/t^{\frac{1}{2}}$ is bounded both above and below:

$$k t^{\frac{1}{2}} \leqq M \leqq k t^{\frac{1}{2}}. \tag{13}$$

If we use best possible constants in (10) and (11), we can obtain

$$b_n(2c_1nt)^{\frac{1}{2}} \leqq M \leqq (2c_2nt)^{\frac{1}{2}}[1+\min(\lambda, (\lambda/2)^{\frac{1}{2}})],$$

where

$$b_n = (n/2t)^{\frac{1}{2}}\{\pi^{\frac{1}{2}}/[\tfrac{1}{2}(n-1)]!\}^{1/n} \geqq 2^{-1/2n}$$

and

$$\lambda = \tfrac{1}{2}\log(c_2/c_1) - \log b_n \leqq (1/2n)\log 2 + \tfrac{1}{2}\log(c_2/c_1).$$

Thus λ is relatively small. Since $b_n \to 1$ as $n \to \infty$, the bounds sharpen with increasing n; indeed, they seem surprisingly sharp. For comparison, $M = (2nct)^{\frac{1}{2}}$ in the simple heat equation where $C_{ij} = c\delta_{ij}$ and $c_1 = c_2 = c$.

Part II: The G Bound.

Here we obtain a result limiting the extent to which a fundamental solution can be very small over a large volume of space near its source point. From this result, we can show there is some overlap, defined as $\int \min(T_1, T_2)dx$, of two fundamental solutions with nearby source points, starting simultaneously.

Let T be $S(x, t, 0, 0)$ and let

$$U(\xi, t) = t^{n/2}T(t^{\frac{1}{2}}\xi, t). \tag{14}$$

This coordinate transformation and renormalization makes $\int U\, d\xi = 1$, where $d\xi$ is the volume element. Furthermore, if μ is the constant such that $M \leqq \mu t^{\frac{1}{2}}$, we have $\int |\xi| U\, d\xi \leqq \mu$. For U, equation (1) transforms to

$$2tU_t = nU + \xi\cdot\nabla U + 2\nabla\cdot(C\cdot\nabla U). \tag{15}$$

Let

$$G = \int \exp(-|\xi|^2)\log(U+\delta)\, d\xi, \tag{16}$$

where δ is a small positive constant. G is sensitive to areas where $|\xi|$ is not large and U is small. These tend to make G strongly negative. We later obtain a lower bound on G of the form

$$G \geqq -k(-\log\delta)^{\frac{1}{2}},$$

valid for sufficiently small δ. This bound limits the possibility for U to be small in a large portion of the region where $|\xi|$ is not large. From $U > 0$ the weak lower bound $G > \pi^{n/2}\log\delta$ follows immediately.

Differentiating (16) with respect to time and using (15), we obtain

$$2tG_t = H_1 + H_2 + H_3,$$

where

$$H_1 = n\int \exp(-|\xi|^2)U/(U+\delta)\,d\xi \geqq 0,$$

$$H_2 = \int \exp(-|\xi|^2)\xi\cdot\nabla\log(U+\delta)\,d\xi$$
$$= -\int \nabla\cdot[\exp(-|\xi|^2)\xi]\log(U+\delta)\,d\xi,$$

by integration by parts, so that

$$H_2 = -\int \exp(-|\xi|^2)(\nabla\cdot\xi)\log(U+\delta)\,d\xi$$
$$+\int \exp(-|\xi|^2)(2|\xi|\nabla|\xi|)\cdot\xi\log(U+\delta)\,d\xi$$
$$= -nG + 2\int \exp(-|\xi|^2)|\xi|^2[\log\delta + \log(1+U/\delta)]\,d\xi,$$

hence,

$$H_2 \geqq -nG + 2\log\delta\int|\xi|^2\exp(-|\xi|^2)\,d\xi \geqq -nG + n\pi^{n/2}\log\delta;$$

finally

$$H_3 = 2\int \exp(-|\xi|^2)\nabla\cdot(C\cdot\nabla U)/(U+\delta)\,d\xi$$
$$= -2\int \nabla[\exp(-|\xi|^2)/(U+\delta)]\cdot C\cdot\nabla U\,d\xi$$
$$= 4\int(\exp(-|\xi|^2)[|\xi|\nabla|\xi|\cdot C\cdot\nabla U]/(U+\delta)\,d\xi$$
$$+ 2\int \exp(-|\xi|^2)[\nabla U\cdot C\cdot\nabla U]/(U+\delta)^2\,d\xi$$
$$= H_3' + H_3'',$$

where

$$H_3' = 4\int \exp(-|\xi|^2)\xi\cdot C\cdot\nabla\log(U+\delta)\,d\xi,$$

$$H_3'' = 2\int \exp(-|\xi|^2)\nabla\log(U+\delta)\cdot C\cdot\nabla\log(U+\delta)\,d\xi.$$

From the Schwarz inequality,

$$(H_3')^2 \leqq \{4\int \exp(-|\xi|^2)\xi\cdot C\cdot\xi\,d\xi\}$$
$$\times\{4\int \exp(-|\xi|^2)\nabla\log(U+\delta)\cdot C\cdot\nabla\log(U+\delta)\,d\xi\}$$
$$\leqq \{4c_2\int|\xi|^2\exp(-|\xi|^2)\,d\xi\}2H_3''$$
$$\leqq (4c_2\cdot\tfrac{1}{2}n\pi^{\frac{1}{2}n})\cdot 2H_3'' = 4nc_2\pi^{\frac{1}{2}n}H_3''.$$

Hence

$$|H_3'| \leqq k(H_3'')^{\frac{1}{2}}.$$

Furthermore,

$$H_3'' \geqq 2c_1\int \exp(-|\xi|^2)|\nabla\log(U+\delta)|^2\,d\xi \tag{17}$$

Combining the lower bounds available for H_1, H_2, and H_3', we obtain

$$(18)\qquad \begin{aligned} 2tG_t &\geqq H_1 + H_2 + H_3'' - |H_3'| \\ &\geqq 0 + (-nG + n\pi^{\frac{1}{2}n}\log\delta) + H_3'' - k(H_3'')^{\frac{1}{2}} \\ &\geqq k\log\delta - nG - k(H_3'')^{\frac{1}{2}} + H_3''. \end{aligned}$$

When we bound H_3'' below in terms of G, (18) will yield a lower bound on G.

A function $f(\xi) = f(\xi_1, \xi_2, \cdots, \xi_n)$ may be expanded in products of Hermite polynomials, of the form $\prod H_{\nu(i)}(\xi_i)$, where the polynomials are defined and orthonormalized so that $\int_{-\infty}^{+\infty} \exp(-s^2) H_\nu(s) H_\lambda(s)\, ds = \delta_{\nu\lambda}$. The identity $dH_\nu(s)/ds = (2\nu)^{\frac{1}{2}} H_{\nu-1}(s)$ obtains, and the coefficients of these products in the similar expansion of, say, $\partial f/\partial\xi_i$ depend very simply on the coefficients in the expansion of f. If $\int \exp(-|\xi|^2) f\, d\xi = 0$, the coefficient of $\prod H_0(\xi_i)$ is zero and we obtain

$$\int \exp(-|\xi|^2)|\nabla f|^2\, d\xi = \sum \int \exp(-|\xi|^2)(\partial f/\partial\xi_i)^2 d\xi \geqq 2\int \exp(-|\xi|^2) f^2\, d\xi.$$

Applying the above, with $f = \log(U+\delta) - \pi^{-n/2}G$, to (17), we obtain

$$(19)\qquad H_3'' \geqq 4c_1 \int \exp(-|\xi|^2)[\log(U+\delta) - \pi^{-n/2}G]^2\, d\xi.$$

The quantity $U^{-1}[\log(U+\delta) - \pi^{-n/2}G]^2$, related to the integrand in (19), is large for very small U, then decreases to zero, rises from zero to a local maximum at $U = U_c$, say, and finally decreases monotonically as $U \to \infty$ for $U \geqq U_c$. (We know $\log\delta - \pi^{-n/2}G < 0$.) The equation for the maximum point U_c is $\log(U_c+\delta) - \pi^{-n/2}G = 2U_c/(U_c+\delta)$, from which $U_c < U_0 = \exp(2 + \pi^{-n/2}G)$. Therefore the quantity under discussion is decreasing for $U \geqq U_0$ The bound (8), $T \leqq kt^{-n/2}$, corresponds to $U \leqq k$. Hence the quantity has a lower bound of the form $k[\log(k+\delta) - kG]^2$ for $U \geqq U_0$. Applying this to (19), we may say

$$H_3'' \geqq 4c_1 \int \exp(-|\xi|^2) k[\log(k+\delta) - kG]^2 U^*\, d\xi,$$

where $U^* = U$ for $U > U_0$ and $U^* = 0$ for $U \leqq U_0$. Thus we are ignoring the contribution to (19) of the region where $U \leqq U_0$ and taking the worst case, U as large as possible, in the remaining region. For sufficiently negative G, the expression $\log(k+\delta) - kG$ will remain positive when δ is omitted, so that $[\log k - kG]^2 < [\log(k+\delta) - kG]^2$, and we can simplify the above inequality on H_3'' to the form

$$(20)\qquad H_3'' \geqq (k - kG)^2 \int \exp(-|\xi|^2) U^*\, d\xi.$$

Let $\lambda = \int U^*\, d\xi$ and observe that $\int |\xi|\, U^*\, d\xi \leqq \int |\xi|\, U\, d\xi \leqq \mu$. Therefore

$$\mu \geqq \int_{|\xi| \geqq 2\mu/\lambda} |\xi| U^* \, d\xi \geqq (2\mu/\lambda) \int_{|\xi| \geqq 2\mu/\lambda} U^* \, d\xi,$$

hence,

$$\int_{|\xi| \geqq 2\mu/\lambda} U^* \, d\xi \leqq \tfrac{1}{2}\lambda, \text{ and so } \int_{|\xi| \leqq 2\mu/\lambda} U^* \, d\xi \geqq \lambda - \tfrac{1}{2}\lambda = \tfrac{1}{2}\lambda.$$

This result can be applied to (20) and yields

$$H_3'' \geqq (k - kG)^2 \cdot \exp(-(2\mu/\lambda)^2)(\tfrac{1}{2}\lambda). \tag{21}$$

This is not effective unless we can bound λ below, or bound $\int \hat{U} \, d\xi = 1 - \lambda$ above, where $\hat{U} = U - U^*$ so that $\hat{U} = 0$ unless $U \leqq U_0$, in which case $\hat{U} = U$. Of course, we know $\int |\xi| \hat{U} \, d\xi \leqq \mu$ because $\hat{U} \leqq U$. Under the moment constraint and the constraint $\hat{U} \leqq U_0$, the maximum of $\int \hat{U} \, d\xi$ is clearly realized by having $\hat{U} = U_0$ for $|\xi| \leqq \rho$ and $\hat{U} = 0$ for $|\xi| > \rho$, where ρ is such that

$$\int |\xi| \hat{U} \, d\xi = \int |\xi| U_0 \, d\xi = [n\pi^{n/2}/(n+1)(n/2)!]\rho^{n+1} U_0 = \mu.$$

This makes

$$1 - \lambda = \int \hat{U} \, d\xi = [\pi^{n/2}/(n/2)!]\rho^n U_0,$$

$$1 - \lambda \leqq U_0 (k\mu/U_0)^{n/(n+1)} \text{ or } 1 - \lambda \leqq k U_0^{1/(n+1)}.$$

If U_0, which is $\exp(2 + \pi^{-n/2} G)$, is small enough, then $1 - \lambda$ is small and λ is bounded below. Thus $\lambda \geqq \frac{1}{2}$, say, for all sufficiently large $-G$. Now from (21), we have

$$H_3'' \geqq (k - kG)^2$$

for sufficiently large $-G$.

Returning to inequality (18) controlling G_t and applying the above result, we can state that *for sufficiently negative* G,

$$\begin{aligned} 2dG/d(\log t) = 2tG_t &\geqq -nG + k \log \delta + (k - kG)^2 - k(k - kG) \\ &\geqq k|G|^2 + k \log \delta. \end{aligned} \tag{22}$$

Let $G_1(c_1, c_2, n)$ be the number such that when $G \leqq G_1$, we know G is small enough to make (22) valid. Let $G_2(c_1, c_2, n, \delta) = -k(-\log \delta)^{\frac{1}{2}}$ be the largest number such that $k|G|^2 + k \log \delta > 0$ for all $G < G_2$. Then $\min(G_1, G_2) = G_3$ is the smallest possible value of G. If we had $G(t_1) = G_3 - \epsilon$, we would have $dG/d(\log t) \geqq \epsilon^*$ for all $t \leqq t_1$, and consequently $G(t) \leqq G(t_1) - \epsilon^* \log(t_1/t)$, which implies $G \to -\infty$ as $t \to 0$. But since $G \geqq \pi^{n/2} \log \delta$, the hypothesis $G(t_1) = G_3 - \epsilon$ is impossible. Our conclusion is $G \geqq G_3$, or

$$G \geqq -k(-\log \delta)^{\frac{1}{2}} \tag{23}$$

for all *sufficiently small values of* δ, because $G_2 \leqq G_1$ and

$$G_3 = G_2 = -k(-\log\delta)^{\frac{1}{2}}$$

when δ is small enough.

Part III: The Overlap Estimate

Let T_1 and T_2 be two fundamental solutions $S(x, t, x_1, 0)$ and $S(x, t, x_2, 0)$ with nearby sources. Change coordinates, defining $U_1 = t^{n/2}T_1(t^{\frac{1}{2}}\xi, t)$ and $U_2 = t^{n/2}T_2(t^{\frac{1}{2}}\xi, t)$. Let $\xi_1 = x_1/t^{\frac{1}{2}}$ and $\xi_2 = x_2/t^{\frac{1}{2}}$. Here the source of the (renormalized) fundamental solution U_i is ξ_i rather than the origin, which was the source of U in Part II. Taking this into account, we apply (23), obtaining

$$\int \exp(-|\xi - \xi_i|^2)\log(U_i + \delta)\,d\xi = G_i \geqq -k(-\log\delta)^{\frac{1}{2}},$$

where $i = 1$ or 2 and δ must be sufficiently small. We may add the inequalities above and obtain

$$\int \max_i[\exp(-|\xi - \xi_i|^2)]\max_i[\log(U_i + \delta)]\,d\xi$$
$$+ \int \min_i[\exp(-|\xi - \xi_i|^2)]\min_i[\log(U_i + \delta)]\,d\xi \geqq -2k(-\log\delta)^{\frac{1}{2}}$$

in which we form two integrals with sum at least as large as the sum of the original integrals. We abbreviate the above to

$$\int f^* \log(U_{\max} + \delta)\,d\xi + \int \hat{f}\log(U_{\min} + \delta)\,d\xi \geqq -k(-\log\delta)^{\frac{1}{2}}.$$

For the first integral, we observe (assuming $\delta \leqq 1$)

$$\int f^* \log(U_{\max} + \delta)\,d\xi \leqq \int f^*(U_1 + U_2)\,d\xi \leqq \int (U_1 + U_2)\,d\xi = 2.$$

For the second integral,

$$\int \hat{f}\log(U_{\min} + \delta)\,d\xi \leqq \log\delta \int \hat{f}\,d\xi + \max[\hat{f}]\int \log(1 + U_{\min}/\delta)\,d\xi$$
$$\leqq w\log\delta + \delta^{-1}\int U_{\min}\,d\xi,$$

where

$$w = \int \min[\exp(-|\xi - \xi_1|^2), \exp(-|\xi - \xi_2|^2)]\,d\xi.$$

Therefore we obtain

$$2 + w\log\delta + \delta^{-1}\int \min(U_1, U_2)\,d\xi \geqq -k(-\log\delta)^{\frac{1}{2}},$$

or

$$\int \min(T_1, T_2)\,dx = \int \min(U_1, U_2)\,d\xi \geqq \delta[-2 - w\log\delta - k(-\log\delta)^{\frac{1}{2}}]. \tag{24}$$

This is valid for sufficiently small δ, say for $\delta \leqq \delta_1$. Also, there is a value $\delta_2(w)$ such that for $\delta < \delta_2(w)$, the bracketed expression is positive. If we set $\delta = \frac{1}{2}\min(\delta_1, \delta_2)$, the right member of (24) is definitely positive, and we may conclude

$$(25) \qquad \int \min(T_1, T_2)\, dx \geqq \phi(|\xi_1 - \xi_2|) \geqq \phi(|x_1 - x_2|/t^{\frac{1}{2}})$$

because w is a function of $|\xi_1 - \xi_2|$. The function ϕ is decreasing but always positive. It is an a priori function, determined only by c_1, c_2, and n. This inequality (25) is our first estimate on the overlap of fundamental solutions. Its weakness is that we know little about the function ϕ.

Part IV: Continuity in Space.

We can obtain a stronger inequality by iterative use of (25). Observe that

$$(26) \qquad \begin{aligned} \tfrac{1}{2}\int |T_1 - T_2|\, dx &= \tfrac{1}{2}\int [T_1 + T_2 - 2\min(T_1, T_2)]\, dx \\ &\leqq 1 - \phi(|x_1 - x_2|/t^{\frac{1}{2}}) = \psi(|x_1 - x_2|/t^{\frac{1}{2}}) \end{aligned}$$

in which we define the function ψ, which is increasing but always less than one.

Let $T_a = \max(T_1 - T_2, 0)$ and $T_b = \max(T_2 - T_1, 0)$ so that $T_a + T_b = |T_1 - T_2|$ and $\int (T_a - T_b)\, dx = \int (T_1 - T_2)\, dx = 0$. Then

$$\int T_a dx = \int T_b dx = A(t) = \tfrac{1}{2}\int |T_1 - T_2|\, dx \leqq \psi(|x_1 - x_2|/t^{\frac{1}{2}}),$$

defining $A(t)$. Let

$$\chi(x, \bar{x}, t) = T_a(x) T_b(\bar{x})/A(t).$$

Let $T_a^*(x', t', t)$ be the bounded solution in x' and t' of (1) defined for $t' \geqq t$ and having the initial value $T_a^*(x, t, t) = T_a(x, t)$. Define T_b^* similarly. Then from (4),

$$\begin{aligned} T_a^*(x', t', t) &= \int S(x', t', x, t) T_a(x, t)\, dx \\ &= \int\int S(x', t', x, t) \chi(x, \bar{x}, t)\, dx\, d\bar{x}, \end{aligned}$$

and

$$\begin{aligned} T_1(x', t') - T_2(x', t') &= T_a^* - T_b^* \\ &= \int\int [(x', t', x, t) - S(x', t', \bar{x}, t)] \chi(x, \bar{x}, t)\, dx\, d\bar{x} \end{aligned}$$

by the superposition principle ($T_1 - T_2$ and $T_a^* - T_b^*$ are both solutions of (1) for $t' \geqq t$, and by definition, $T_a^* - T_b^* = T_1 - T_2$ at $t' = t$.). Integrating this over dx', we obtain

$$\tfrac{1}{2}\int |T_1(x',t') - T_2(x',t')|\,dx'$$
$$\leqq \int\int\int |S(x',t',x,t) - S(x',t',\bar{x},t)|\,dx'\,\chi(x,\bar{x},t)\,dx\,d\bar{x},$$

whence

$$A(t') \leqq \int\int \psi(|x-\bar{x}|/(t'-t)^{\frac{1}{2}})\chi(x,\bar{x},t)\,dx\,d\bar{x} \tag{27}$$

by application of (26). Incidentally, the right member above is

$$\leqq \int\int \chi(x,\bar{x},t)\,dx\,d\bar{x} = A(t),$$

thus $A(t') \leqq A(t)$ when $t' \geqq t$. This inequality (27) is the key to the iterative argument which strengthens (25) and (26).

To begin the iterative argument, we choose any specific number d and let $\epsilon = \phi(d) = 1 - \psi(d)$. (If we were trying to get an explicit formula for the exponent α in (2), we would choose d with regard to an explicit formula for $\phi(d)$ so as to optimize the result.) Let $\sigma = 1 - \epsilon/4$. For each integer ν, let t_ν be the time (or the least time) at which $A(t) = A(t_\nu) = \sigma^\nu$, if t_ν exists. This is in reference to a specific pair, T_1 and T_2, of fundamental solutions. We know, for example, that $t_1 < \tau$, where $\tau = |x_1 - x_2|^2/d^2$, because $A(\tau) \leqq \psi(|x_1 - x_2|/\tau^{\frac{1}{2}}) = \psi(d) = 1 - \epsilon$ and $\sigma = 1 - \epsilon/4 > 1 - \epsilon$, so that $A(\tau) < A(t_1) = \sigma$.

Let $M_a(t) = \int |x - x_0|\,T_a dx$, where x_0 is $\frac{1}{2}(x_1 + x_2)$, the midpoint of the line segment joining the source points x_1 and x_2 of the fundamental solutions T_1 and T_2. Define M_b similarly and let $M_\nu = \max[M_a(t_\nu), M_b(t_\nu)]$. We decompose T_a into nearer and farther parts T_a' and $T_a - T_a'$ at each time t_ν as follows: for $|x - x_0| \leqq 2\sigma^{-\nu}M_\nu$, define $T_a' = T_a$; otherwise $T_a' = 0$. Then $2\sigma^{-\nu}M_\nu \int (T_a - T_a')\,dx \leqq \int |x - x_0|\,(T_a - T_a')\,dx \leqq \int |x - x_0|\,T_a dx \leqq M_\nu$, and consequently, $\int (T_a - T_a')\,dx \leqq \frac{1}{2}\sigma^\nu$ and $\int T_a'\,dx \geqq \frac{1}{2}\sigma^\nu$. Define T_b' similarly and define $\chi_\nu'(x,\bar{x}) = \sigma^{-\nu}T_a'(x)T_b'(\bar{x})$. Now, applying (27) with $t = t_\nu$, we can say

$$A(t') \leqq \int\int \psi(|x-\bar{x}|)/(t'-t_\nu)^{\frac{1}{2}}[\{\chi(x,\bar{x},t_\nu) - \chi_\nu'(x,\bar{x})\} + \chi_\nu'(x,\bar{x})]\,dx d\bar{x}$$
$$\leqq \int\int \{\chi - \chi_\nu'\}\,dx d\bar{x} + \psi(4\sigma^{-\nu}M_\nu/(t'-t_\nu)^{\frac{1}{2}}) \int\int \chi_\nu'\,dx d\bar{x},$$

because when $\chi_\nu' > 0$, we know both $T_a' > 0$ and $T_b' > 0$ so that both $|x - x_0|$ and $|\bar{x} - x_0|$ are $\leqq 2\sigma^{-\nu}M_\nu$, and consequently, $|x - \bar{x}| \leqq 4\sigma^{-\nu}M_\nu$, and we also know that $\chi \geqq \chi_\nu'$ and $\psi < 1$. Proceeding further,

$$A(t') \leqq \int\int \chi\,dx d\bar{x} - [1 - \psi(4\sigma^{-\nu}M_\nu/(t'-t_\nu)^{\frac{1}{2}})] \int\int \chi_\nu'\,dx d\bar{x}$$
$$\leqq \sigma^\nu - [1-\psi]\sigma^{-\nu}\int T_a' dx \int T_b' dx \leqq \sigma^\nu - [1-\psi]\sigma^{-\nu}(\sigma^\nu/2)^2$$
$$\leqq \sigma^\nu[3/4 + 1/4\psi(4\sigma^{-\nu}M_\nu/(t'-t_\nu)^{\frac{1}{2}})].$$

We now set $t' = t_\nu + 16\sigma^{-2\nu}(M_\nu)^2 d^{-2}$, and the argument of ψ above becomes d. Then since $\psi(d) = 1 - \epsilon$, we obtain

$$A(t') \leqq \sigma^\nu[3/4 + 1/4(1-\epsilon)] = \sigma^\nu(1 - \epsilon/4) = \sigma^{\nu+1}.$$

Hence

$$t_{\nu+1} \leqq t' = t_\nu + 16\sigma^{-2\nu}(M_\nu)^2 d^{-2}. \tag{28}$$

This will bound the sequence $\{t_\nu\}$ of times after we obtain a bound on the sequence $\{M_\nu\}$ of moments.

Observe that

$$\begin{aligned} T_a(x', t') &= \max(T_1(x', t') - T_2(x', t'), 0) \\ &= \max(T_a^*(x', t', t) - T_b^*(x', t', t), 0) \leqq T_a^*(x', t', t) \\ &= \int S(x', t', x, t) T_a(x, t)\, dx. \end{aligned}$$

Therefore

$$\begin{aligned} M_a(t') &= \int |x' - x_0|\, T_a(x', t')\, dx' \\ &\leqq \int\int [|x' - x| + |x - x_0|] S(x', t', x, t) T_a(x, t)\, dx dx'; \end{aligned}$$

hence

$$\begin{aligned} M_a(t') \leqq{}& \int |x - x_0|\, T_a(x, t) \int S(x', t', x, t)\, dx' dx \\ &+ \int T_a(x, t) \int |x' - x|\, S(x', t', x, t)\, dx' dx, \end{aligned}$$

or

$$\begin{aligned} M_a(t') &\leqq \int |x - x_0|\, T_a(x, t)\, dx + \mu(t' - t)^{\frac{1}{2}} \int T_a(x, t)\, dx \\ &\leqq M_a(t) + A(t)\mu(t' - t)^{\frac{1}{2}}. \end{aligned}$$

Now let t and t' be t_ν and $t_{\nu+1}$, use a similar estimate for M_b and the definition $M_\nu = \max(M_a(t_\nu), M_b(t_\nu))$, and obtain, by (28),

$$\begin{aligned} M_{\nu+1} &\leqq M_\nu + \sigma^\nu \mu (t_{\nu+1} - t_\nu)^{\frac{1}{2}} \\ &\leqq M_\nu + \sigma^\nu \mu (16\sigma^{-2\nu}(M_\nu)^2 d^{-2})^{\frac{1}{2}} \leqq M_\nu(1 + 4\mu/d). \end{aligned}$$

Now $t_0 = 0$ and $M_0 = M_a(t_0) = M_b(t_0) = \frac{1}{2}|x_1 - x_2|$, because T_1 and T_2 concentrate at x_1 and x_2 as $t \to 0$, and $|x_1 - x_0| = |x_2 - x_0| = \frac{1}{2}|x_1 - x_2|$ since $x_0 = \frac{1}{2}(x_1 + x_2)$. Therefore we have

$$M_\nu \leqq \tfrac{1}{2}|x_1 - x_2|(1 + 4\mu/d)^\nu.$$

With this and (28), the sequence $\{t_\nu\}$ can be bounded:

$$t_{\nu+1} \leqq t_\nu + 16\sigma^{-2\nu}[\tfrac{1}{2}|x_1 - x_2|(1 + 4\mu/d)^\nu]^2 d^{-2}.$$

Hence

$$t_{\nu+1} \leqq 4d^{-2} \mid x_1 - x_2 \mid^2 \sum_{\lambda=0}^{\nu} [(1 + 4\mu/d)/\sigma)]^{2\lambda}.$$

Summing this geometrical series,

$$t_\nu / \mid x_1 - x_2 \mid^2 \leqq 4d^{-2}\{(\sigma^{-2}(1 + 4\mu/d))^{2\nu+2}/[\sigma^{-2}(1 + 4\mu/d) - 1]\} \equiv \zeta\eta^\nu$$

(definition of ζ, η). Now for any time t, define $\nu(t)$ to be either zero or the integer such that

$$\zeta\eta^{\nu(t)} \leqq t/\mid x_1 - x_2 \mid^2 < \zeta\eta^{\nu(t)+1}$$

if this integer exists. Then $t_{\nu(t)} \leqq t$ and $A(t) \leqq A(t_{\nu(t)}) = \sigma^{\nu(t)}$. Also,

$$\nu(t) \geqq (\log(t/\zeta \mid x_1 - x_2 \mid^2)/\log\eta) - 1.$$

From these observations, we conclude

$$\sigma^{\nu(t)} \leqq \sigma^{-1} \exp[(\log\sigma/\log\eta)\log(t/\zeta \mid x_1 - x_2 \mid^2)];$$

hence

$$A(t) \leqq \sigma^{-1}(t/\zeta \mid x_1 - x_2 \mid^2)^{\log\sigma/\log\eta},$$

or

$$\tfrac{1}{2}\int \mid T_1 - T_2 \mid dx \leqq \sigma^{-1}\zeta^{\alpha/2}(\mid x_1 - x_2 \mid/t^{\frac{1}{2}})^\alpha,$$

where $\frac{1}{2}\alpha = -\log\sigma/\log\eta$.

Both σ and η are determined by d. Specifically, $\sigma = 1 - \frac{1}{4}\phi(d)$ and $\eta = [\sigma^{-2}(1 + 4\mu/d)]^2$. An optimal choice of d in relation to $\phi(d)$ would maximize α. We may choose d arbitrarily as, $d^2 = c_1$, say; this will make α a function of μ and c_2/c_1 (proof omitted). In any case, even if we set $d = 1$, we obtain the estimate

$$(29) \qquad \int \mid S(x, t, x_1, t_0) - S(x, t, x_2, t_0) \mid \leqq A_1(\mid x_1 - x_2 \mid/(t - t_0)^{\frac{1}{2}})^\alpha,$$

where A_1 and α are a priori constants depending only on n, c_1 and c_2. Also, for the dual adjoint equation,

$$(30) \qquad \int \mid S(x_1, t, x_0, t_0) - S(x_2, t, x_0, t_0) \mid dx_0 \leqq A_1(\mid x_1 - x_2 \mid/(t - t_0)^{\frac{1}{2}})^\alpha.$$

With (30), we obtain the estimate for the continuity in space of a bounded solution of (1). If $T(x, t)$ satisfies (1) and $\mid T \mid \leqq B$ for $t \geqq t_0$, then

$$\begin{aligned} \mid T(x_1, t) - T(x_2, t) \mid &\leqq \mid \int [S(x_1, t, x_0, t) - S(x_2, t, x_0, t_0)] T(x_0, t_0)\, dx_0 \mid \\ &\leqq B \int \mid S(x_1, t, x_0, t_0) - S(x_2, t, x_0, t_0) \mid dx_0. \end{aligned}$$

Hence,

$$(31) \qquad \mid T(x_1, t) - T(x_2, t) \mid \leqq BA_1(\mid x_1 - x_2 \mid/(t - t_0)^{\frac{1}{2}})^\alpha.$$

Part V: Time Continuity.

(31) gives half of (2); the remaining part, time continuity, can be derived from (31) and the moment bound (13). Let $T(x,t)$ be a solution of (1) with $|T| \leqq B$ for $t \geqq t_0$. Then for $t' > t > t_0$ we have

$$T(x,t) - T(x,t') = T(x,t) - \int S(x,t',\bar{x},t)T(\bar{x},t)\,d\bar{x}$$
$$= \int S(x,t',\bar{x},t)[T(x,t) - T(\bar{x},t)]\,d\bar{x},$$

since $\int S\,d\bar{x} = 1$. Therefore, $|T(x,t) - T(x,t')| \leqq$

$$\int S(x,t',\bar{x},t)\,|T(x,t) - T(\bar{x},t)|\,d\bar{x}$$
$$\leqq \int S(x,t',x+y,t)\,|T(x,t) - T(x+y,t)|\,dy.$$

Now we separate this integral into two parts, in terms of a radius ρ; one where $|y| \leqq \rho$ and one where $|y| > \rho$. Thus $|T(x,t) - T(x,t')| \leqq I_1 + I_2$, where

$$I_1 = \int_{|y| \leqq \rho} S(x,t',x+y,t)\,|T(x,t) - T(x+y,\,t)|\,dy \leqq BA_1(\rho/(t-t_0)^{\frac{1}{2}})^{\alpha}$$

(because $\int S\,dy = 1$), and

$$I_2 = \int_{|y|>\rho} S(x,t',x+y,t)\,|T(x,t) - T(x+y,t)|\,dy$$
$$\leqq 2B\rho^{-1}\int_{|y|>\rho} |y|\,S(x,t',x+y,t)\,dy \leqq 2B\mu(t'-t)^{\frac{1}{2}}/\rho.$$

Adding the two inequalities,

$$|T(x,t) - T(x,t')| \leqq BA_1(\rho/(t-t_0)^{\frac{1}{2}})^{\alpha} + 2B\mu(t'-t)^{\frac{1}{2}}/\rho,$$

and if we choose ρ so as to minimize the sum, then

$$\alpha A_1\rho^{1+\alpha} = 2\mu(t'-t)^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}\alpha},$$

and we obtain

$$(32) \qquad |T(x,t) - T(x,t')| \leqq BA_2[(t'-t)/(t-t_0)]^{\frac{1}{2}\alpha/(1+\alpha)},$$

where $A_2 = (1+\alpha)A_1(2\mu/\alpha A_1)^{\alpha/(1+\alpha)}$. This result (32), combined with (31) yields (2), with $A = \max(A_1, A_2)$.

Part VI: Elliptic Problems.

We treat elliptic problems as a special type of parabolic problem, one in which the coefficients of the equation are time independent and a time independent solution is sought. The Hölder continuity of solutions of uniformly elliptic equations of the form $\nabla \cdot (C \cdot \nabla T) = 0$ appears as a corollary of the result for the parabolic case. There may exist another proof of our result (3). P. R. Garabedian writes from London of a manuscript by Ennio de Giorgi containing such a result. See de Giorgi's note, reference [9].

Let $\mathcal{D}$ be a domain in space-time defined by the constraints $|x| \leqq \sigma$ and $t \geqq 0$. Then $\mathcal{D}$ is a solid semi-infinite spherical cylinder. Call $\mathcal{B}$ the points of the cylindrical surface or boundary of $\mathcal{D}$, where $|x| = \sigma$. Let $\mathcal{D}_0$ be the points of the base of $\mathcal{D}$, where $t = 0$. Define $\mathcal{B}^*$ as the total boundary of $\mathcal{D}$, the union $\mathcal{B} \cup \mathcal{D}_0$, of the base and cylindrical surfaces.

A "Dirichlet parabolic boundary value problem" is given when values of T are specified on $\mathcal{B}^*$ and we ask for a solution of (1) in $\mathcal{D}$ assuming these specified values on $\mathcal{B}$. The solution of the problem must depend linearly on the boundary values; also, the maximum and minimum principles must hold. These facts require that the solution $T(x, t)$ be determined in this way:

$$T(x, t) = \int T(\xi)\, d\rho(\xi; x, t). \tag{33}$$

Here (x, t) is a point of $\mathcal{D}$, ξ is any point of $\mathcal{B}^*$, and $d\rho(\xi; x, t)$ is a positive measure, associated with ξ, which has $\int d\rho = 1$ and which vanishes for $t(\xi) > t$. The time and space coordinates of the point ξ are called $t(\xi)$ and $x(\xi)$. We cannot pause here for a detailed justification of (33), but refer the reader to the literature.

We can define a boundary value problem for which we know the solution in advance by setting $T(\xi) = S(x(\xi), t(\xi), x_0, t_0)$ if $t_0 < 0$. Then the solution of the problem is $S(x, t, x_0, t_0)$, and from (33),

$$S(x, t, x_0, t_0) = \int S(x(\xi), t(\xi), x_0, t_0)\, d\rho(\xi; x, t). \tag{34}$$

This is a powerful identity; it enables us to convert information on fundamental solutions into information on $d\rho$, and in particular, we can obtain a moment bound for $d\rho$. Multiplying (34) by $|x - x_0|$ and integrating, we have

$$\int |x - x_0|\, S(x, t, x_0, t_0)\, dx_0 = \int\int |x - x_0|\, S(x(\xi), t(\xi), x_0, t_0)\, d\rho\, dx_0.$$

Hence

$$\mu(t-t_0)^{\frac{1}{2}} \geqq \int\int \{|x-x(\xi)|-|x_0-x(\xi)|\} S(x(\xi), t(\xi), x_0, t\ ) d\rho\, dx_0,$$

so that

$$\mu(t-t_0)^{\frac{1}{2}} + \int\int |x_0-x(\xi)|\, S(x(\xi), t(\xi), x_0, t_0)\, dx_0 d\rho$$
$$\geqq \int |x-x(\xi)| \int S(x(\xi), t(\xi), x_0, t_0)\, dx_0 d\rho.$$

Since $\int S\, dx_0 = 1$, and from the moment bound (13) again, we obtain

$$\mu(t-t_0)^{\frac{1}{2}} + \int \mu(t(\xi)-t_0)^{\frac{1}{2}} d\rho \geqq \int |x-x(\xi)|\, d\rho.$$

Now $d\rho$ vanishes unless $t(\xi) \leqq t$, and t_0 can be as near to zero as desired; also, $\int d\rho = 1$. Hence we can simplify the above to

$$2\mu t^{\frac{1}{2}} \geqq \int |x-x(\xi)|\, d\rho(\xi; x, t). \tag{35}$$

This moment bound (35) on $d\rho$ enables us to control the relative sizes of the effects of the two parts of the boundary in determining $T(x,t)$, where (x,t) is in $\mathfrak{D}$. Thus

$$\int |x-x(\xi)|\, d\rho \geqq \int \big||x|-|x(\xi)|\big|\, d\rho \geqq (\sigma-|x|) \int_{\mathfrak{B}} d\rho.$$

Hence

$$\int_{\mathfrak{B}} d\rho(\xi; x, t) \leqq 2\mu t^{\frac{1}{2}}/(\sigma-|x|). \tag{36}$$

Now let $T(x)$ be a solution in a region $\mathfrak{R}$ of n-space of $\nabla\cdot(C(x)\cdot\nabla T) = 0$, where $C(x)$ satisfies the uniform ellipticity condition with bounds c_1 and c_2. If we introduce time and define $T(x,t) = T(x)$, then $T(x,t)$ satisfies $\nabla\cdot(C\cdot\nabla T) = T_t$, which is of our form (1). Suppose x_1 and x_2 are two points of $\mathfrak{R}$ and let $d(x_1, x_2)$ be the smaller of $d(x_1)$ and $d(x_2)$, the distances from the boundary of $\mathfrak{R}$ of x_1 and x_2 (of course, $d(x_1, x_2)$ may be infinite). For any $\sigma < d(x_1, x_2)$, we can define $\mathfrak{D}_1$ as the set of points (x,t) in space-time where $|x-x_1| \leqq \sigma$ and $t \geqq 0$; also, $\mathfrak{D}_2$ can be defined for x_2, and the boundaries $\mathfrak{B}_1$, $\mathfrak{B}_2$, etc. can be defined in the obvious way. $T(x,t)$ can be regarded as a solution of a parabolic boundary value problem either in $\mathfrak{D}_1$ or $\mathfrak{D}_2$. Another problem with solution $T'(x,t)$ can be defined at first as an initial value problem in all space by setting $T'(x,0) = T(x)$ for all x where $\min(|x-x_1|, |x-x_2|) \leqq \sigma$, that is, $T'(x,t) = T(x)$ when $(x,t) \in \mathfrak{D}_{10} \cup \mathfrak{D}_{20}$, and setting $T'(x,0) = 0$ for all other x values. If $B(\sigma) = \max|T(x)|$ over the set of x values where $\min(|x-x_1|, |x-x_2|) \leqq \sigma$ then $|T'(x,0)| \leqq B(\sigma)$; furthermore, the solution $T'(x,t)$ satisfies $|T'|$

$\leqq B(\sigma)$ for all $t \geqq 0$ by the maximum principle. We can also regard $T'(x, t)$ as a solution of a boundary value problem, either in $\mathfrak{D}_1$ or in $\mathfrak{D}_2$, where the boundary values are just the values $T'(x(\xi), t(\xi))$ assumed there anyway.

By (33), for any $(x, t) \in \mathfrak{D}_i$,

$$T(x, t) - T'(x, t) = \int [T(x(\xi), t(\xi)) - T'(x(\xi), t(\xi))] d\rho_i(\xi; x, t),$$

where $d\rho_i$ is the measure associated with $\mathfrak{D}_i$, and $i = 1, 2$. Now $T(x, t) = T(x)$ is time independent, and on $\mathfrak{D}_{i0}$ we have $T(x, t) = T'(x, t) = T(x)$. Therefore,

$$| T(x) - T'(x, t) | \leqq \int_{\mathfrak{B}_i} | T(x(\xi)) - T'(x(\xi), t(\xi)) | \, d\rho_i \leqq 2B(\sigma) \int_{\mathfrak{B}_i} d\rho_i,$$

and

$$| T(x_i) - T((x_i, t) | \leqq 4B(\sigma)\mu t^{\frac{1}{2}}/\sigma,$$

by use of (36). With our Hölder continuity estimate (2) for solutions of $\nabla \cdot (C \cdot \nabla T) = T_t$ in free space, we can bound $| T'(x_1, t) - T'(x_2, t) |$. This, with the inequality above yields

$$| T(x_1) - T(x_2) | \leqq B(\sigma) A (| x_1 - x_2 |/t^{\frac{1}{2}})^{\alpha} + 8\mu B(\sigma) t^{\frac{1}{2}}/\sigma,$$

valid for any positive t. Choice of the optimal t value gives an inequality of the form

$$(37) \qquad | T(x_1) - T(x_2) | \leqq B(\sigma) A' (| x_1 - x_2 |/\sigma)^{\alpha/(\alpha+1)}.$$

If $| T(x) | \leqq B$ in $\mathfrak{R}$, we may set $\sigma = d(x_1, x_2)$ and obtain (3).

Appendix.

The methods used above can give more explicit results, such as an explicit lower bound for the Hölder exponent α. This takes the form $\alpha = \exp[-a_n(\mu^2/c_1)^{n+1}]$, where a_n depends only on the dimension n. However, a sharper estimate for α might take a quite different form. Numerical calculation of extremal examples might give a better picture.

The moment bound (13) serves to control the rate of dispersal of fundamental solutions. An iterative argument based on (33) and (35) obtains stronger results from (13). In this argument, a fundamental solution is treated as the solution of an array of parabolic boundary value problems, the boundaries being a sequence of spheres centered at the source of the fundamental solution. The result is as follows: let $\nu = [\rho/2\mu(t_2 - t_1)^{\frac{1}{2}}]$, the largest integer not greater than $\rho/2\mu(t_2 - t_1)^{\frac{1}{2}}$, then

$$\int_{|x_2-x_1|\geqq\rho} S(x_2, t_2, x_1, t_1)\,dx_2 \leqq (\pi/4)^{\nu/2}/(\nu/2)\,! \leqq \exp[-\tfrac{1}{2}(\nu+1)\log(2(\nu+1)/\pi e)].$$

Hence,

$$(38)\qquad \int_{|x_2-x_1|\geqq\rho} S(x_2, t_2, x_1, t_1)\,dx_2 \leqq \exp\{-\rho\log(\rho/\pi e\mu(t_2-t_1)^{\frac{1}{2}})/4\mu(t_2-t_1)^{\frac{1}{2}}\}.$$

With (38), the reproductive identity (5), and the bound (7), we obtain a pointwise upper bound of the form

$$(39)\qquad S(x_2, t_2, x_1, t_1) \leqq k(t_2-t_1)^{-n/2}\exp[-k\,|\,x_1-x_2\,|\,(t_2-t_1)^{-\frac{1}{2}}\log(k\,|\,x_1-x_2\,|(t_2-t_1)^{-\frac{1}{2}})].$$

On the other hand, we obtain from (5) and (23) (or alternatively, from (38) and an analogue of (25)), by an argument resembling that which gave (25), the lower bound

$$(40)\qquad S(x_2, t_2, x_1, t_1) \geqq (t_2-t_1)^{-n/2}\phi^*(|\,x_1-x_2\,|/(t_2-t_1)^{\frac{1}{2}}),$$

where ϕ^* is an a priori function determined by c_1, c_2, and n. The inequality $S(x_2, t_2, x_1, t_1) \geqq P_aP_bP_c$, where

$$P_a = \min S(x_2, t_2, \bar{x}, \tfrac{1}{2}(t_1+t_2)) \text{ for } |\,\bar{x}-x_1\,| \leqq \rho,$$
$$P_b = \min S(\bar{x}, \tfrac{1}{2}(t_1+t_2), x_1, t_1) \text{ for } |\,\bar{x}-x_2\,| \leqq \rho,$$
$$P_c = \int d\bar{x}, \text{ where } |\,\bar{x}-x_1\,| \leqq \rho \text{ and } |\,\bar{x}-x_2\,| \leqq \rho,$$

can be used in a iterative argument to strengthen (40). For any $\epsilon > 0$, we obtain

$$(41)\qquad S(x_2, t_2, x_1, t_1) \geqq k_1(t_2-t_1)^{-n/2}\exp[-k_2(|\,x_1-x_2\,|/(t_2-t_1)^{\frac{1}{2}})^{2+\epsilon}],$$

where k_1 and k_2 depend on ϵ (and on c_1, c_2, and n).

With (38), (41) and (35), we can estimate the speed of convergence to assigned boundary values of the solution of an elliptic boundary value problem, provided the boundary is "tame" enough. A point ξ on the boundary $\mathcal{B}$ of a region $\mathcal{R}$ is called *regular* if there are two positive numbers ρ and ϵ such that any sphere with radius $\leqq\rho$ and centered at ξ has at least the fraction ϵ of its volume *not* within $\mathcal{B}$. Then there are constants D, σ, and β determined by ϵ, c_2/c_1, and n such that for any x in $\mathcal{R}$ with $|\,x-\xi\,| \leqq \sigma\rho$, we have

$$(42)\qquad \begin{aligned} T(x) &\geqq \min T(\bar{\xi}) - D\,|(x-\xi)/\rho\,|^{\beta}, \\ T(x) &\leqq \max T(\bar{\xi}) + D\,|(x-\xi)/\rho\,|^{\beta}, \text{ where } |\bar{\xi}-\xi| \leqq \rho \end{aligned}$$

($\bar{\xi}$ represents a variable point on the boundary $\mathcal{B}$).

From (42), it follows that the solution of an elliptic boundary value problem is continuous at the boundary if continuous values were assigned on the boundary and all boundary points are regular. With Hölder continuous boundary values, the solution is Hölder continuous in the region and at the boundary.

From the estimates above, we can fairly easily derive a "Harnack inequality" for parabolic equations:

$$(43)\qquad T(x_2, t) \geqq F(T(x_1, t)/B, |x_1 - x_2|/(t-t_0)^{\frac{1}{2}}),$$

provided $0 \leqq T \leqq B$ for $t \geqq t_0$. F is an a priori function, determined by c_1, c_2 and n. For the elliptic case where T is non-negative in a sphere of radius r centered at the origin, the result takes the form

$$(44)\qquad |\log(T(x')/T(x))| \leqq H(r[r - \max(|x|, |x'|)]^{-1}, |x - x'|/r).$$

The a priori function H is determined by c_2/c_1 and n. This result is less easily obtained than (43).

Parabolic or elliptic problems with Neumann boundary conditions can aparently be handled by a relatively straightforward rederivation of the estimates of this paper in the context of the Neumann boundary, obtaining ultimately the Hölder continuity of the solution for any typical boundary shape.

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