Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{ccc} u \le -7 & \text{and} & u \ge 3 \\ \left(-\infty, -7\right] & \text{and} & \left[3, \infty\right) \end{array}}$$

---

2. Solve the following inequality.

$$x^2 + 8x + 12 < 0$$

Step 1
The first thing we need to do is get a zero on one side of the inequality (which is already done for this problem) and then, if possible, factor the polynomial.

6a)

$$\left(x+6\right)\left(x+2\right) < 0$$

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

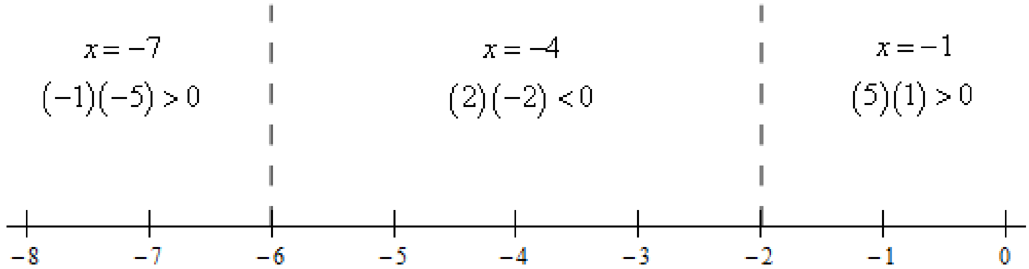
$$x = -6 \qquad\qquad x = -2$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

Hint : Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3
Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c}-6 < x < -2 \\ (-6,-2)\end{array}}$$

---

3. Solve the following inequality.

$$4t^2 \le 15 - 17t$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$4t^2 + 17t - 15 \le 0$$
$$(t+5)(4t-3) \le 0$$

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. From the factored from we can quickly see that the polynomial will be zero at,

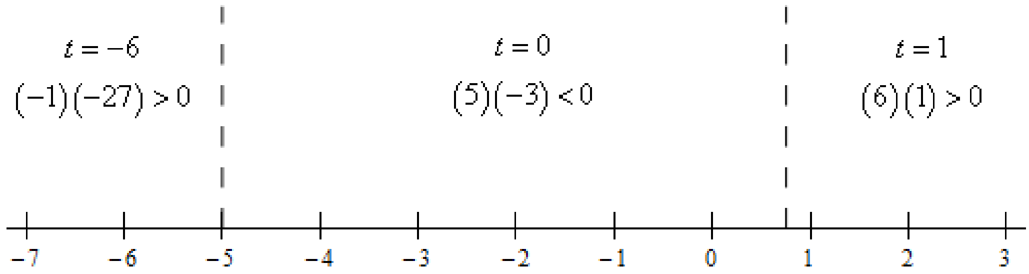
$$t = -5 \qquad\qquad t = \frac{3}{4}$$

Remember that these points are important because they are the only places where the polynomial on the left side of the inequality might change sign. Given that we want to know where the polynomial is zero (which we now know) or negative knowing where the polynomial might change sign will help considerably with determining the answer we're looking for.

Hint : Knowing that the polynomial can only change sign at the points above how can we quickly determine if the polynomial is positive or negative in the ranges between those points?

Step 3
Recall from the discussion in the notes for this section that because the points from the previous step are the only places where the polynomial might change sign we can quickly determine if the polynomial is positive/negative in the ranges between each of these points simply by plugging in "test points" from each region into the polynomial to check the sign.

So, let's sketch a quick number line with the points where the polynomial is zero graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c} -5 \le t \le \dfrac{3}{4} \\ \left[-5, \dfrac{3}{4}\right] \end{array}}$$

---

4. Solve the following inequality.

$$z^2 + 34 > 12z$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the polynomial.

$$z^2 - 12z + 34 > 0$$

In this case the polynomial doesn't factor.

Hint : Where are the only places where the polynomial might change signs?

Step 2
Despite the fact that this is an inequality we first need to know where the polynomial is zero. Because the polynomial didn't factor we'll need to use the quadratic formula to determine where it's zero.

Algebra III MSCCR Standard 17: Solve polynomial and rational inequalities. Relate results to the behavior of the graphs.

6.6 Solving Polynomial and Rational Inequalities

LEARNING OBJECTIVES

1. Solve polynomial inequalities.
2. Solve rational inequalities.

Solving Polynomial Inequalities

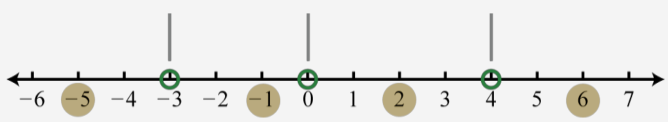
A polynomial inequality is a mathematical statement that relates a polynomial expression as either less than or greater than another. We can use sign charts to solve polynomial inequalities with one variable.

Example 1

$x(x+3)^2(x-4)<0$.

Solution:

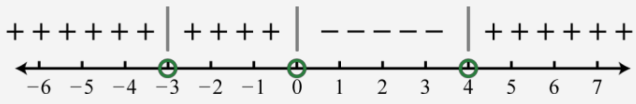
Begin by finding the critical numbers. For a polynomial inequality in standard form, with zero on one side, the critical numbers are the roots. Because $f(x)=x(x+3)^2(x-4)$ is given in its factored form the roots are apparent. Here the roots are: 0, −3, and 4. Because of the strict inequality, plot them using open dots on a number line.



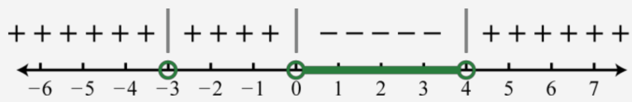
In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if f is positive or negative. Here we choose test values −5, −1, 2, and 6. Remember that we are only concerned with the sign (+ or −) of the result.

$$f(-5)=(-5)(-5+3)^2(-5-4)=(-)(-)^2(-)=+Positive$$
$$f(-1)=(-1)(-1+3)2(-1-4)=(-)(+)^2(-)=+Positive$$
$$f(2)=(2)(2+3)2(2-4)=(+)(+)^2(-)=-Negative$$
$$f(6)=(6)(6+3)2(6-4)=(+)(+)^2(+)=+Positive$$

After testing values we can complete a sign chart.



The question asks us to find the values where $f(x)<0$, or where the function is negative. From the sign chart we can see that the function is negative for *x*-values in between 0 and 4.



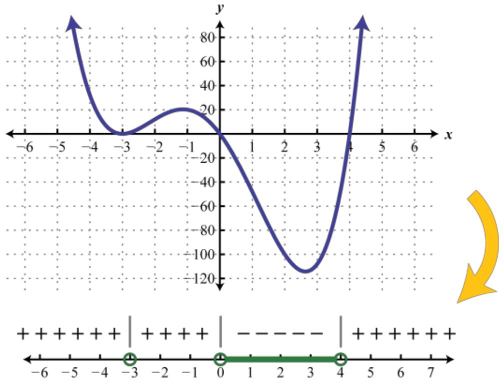
We can express this solution set in two ways:

$$\{x \mid 0<x<4\}\ Set\ notation$$
$$(0,4)\ Interval\ notation$$

In this textbook we will continue to present solution sets using interval notation.

Answer: $(0,4)$

Graphing polynomials such as the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.



Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

Example 2

Solve: $2x^4 > 3x^3 + 9x^2$.

Solution:

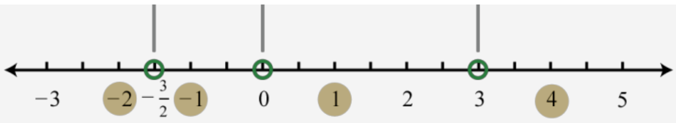
Step 1: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard from.

$$2x^4 > 3x^3 + 9x^2$$
$$2x^4 - 3x^3 - 9x^2 > 0$$

Step 2: Find the critical numbers. Here we can find the zeros by factoring.

$$2x^4 - 3x^3 - 9x^2 = 0$$
$$x^2\left(2x^2 - 3x - 9\right) = 0$$
$$x^2\left(2x + 3\right)\left(x - 3\right) = 0$$

There are three solutions, hence, three critical numbers $-\frac{3}{2}$, 0, and 3.The strict inequality indicates that we should use open dots.

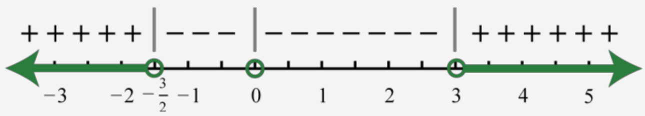


Step 3: Create a sign chart. In this case use $f(x) = x^2(2x+3)(x-3)$ and test values −2, −1, 1, and 4 to determine the sign of the function in each interval.

$$f(-2) = (-2)^2[2(-2)+3](-2-3) = (-)^2(-)(-) = +$$
$$f(-1) = (-1)^2[2(-1)+3](-1-3) = (-)^2(+)(-) = -$$
$$f(1) = (1)^2[2(1)+3](1-3) = (+)^2(+)(-) = -$$
$$f(4) = (4)^2[2(4)+3](4-3) = (+)^2(+)(+) = +$$

With this information we can complete the sign chart.

Step 4: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.



Answer: $(-\infty, -32) \cup (3, \infty)$

Example 3

Solve: $x^3 + x^2 \leq 4(x+1)$.

Solution:

Begin by rewriting the inequality in standard form, with zero on one side.

135 NOTES ON RATIONAL INEQUALITIES

2a)

Solving a rational inequality: Problem type 1

Solve the following inequality.

$$\frac{x-7}{x-2} \leq 0$$

Write your answer using interval notation.

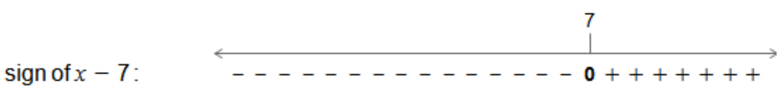
EXAMPLES

We need to find all the values of x that make the quotient $\frac{x-7}{x-2}$ *negative or equal to 0.*
To find these values, we do a sign analysis.

We first look at the sign of the numerator $x-7$.

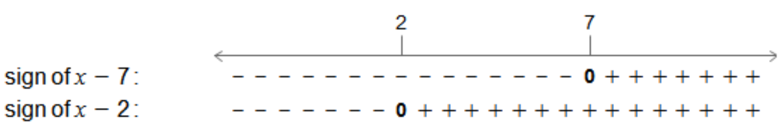
If $x=7$, then $x-7$ is 0 .
If $x<7$, then $x-7$ is negative.
If $x>7$, then $x-7$ is positive.

We show this on the number line.

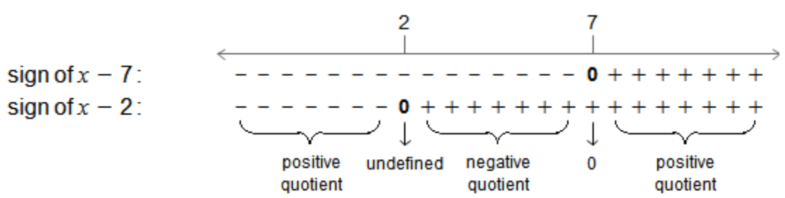


We then look at the sign of the denominator $x-2$.

If $x=2$, then $x-2$ is 0 .
If $x<2$, then $x-2$ is negative.
If $x>2$, then $x-2$ is positive.



Finally, to find the sign of $\frac{x-7}{x-2}$, we use the rules for dividing signed numbers.



So, we have $\frac{x-7}{x-2} \leq 0$ when $2<x$ and $x \leq 7$.

Note that for $x=2$, the quotient is undefined and so 2 *is not* part of the solution.
For $x=7$, the quotient is 0 and so 7 *is* part of the solution.
We write the solution in interval notation.

$(2,7]$

2b

Solving a rational inequality: Problem type 1

Solve the following inequality.

$$\frac{x+1}{-x-6} > 0$$

Write your answer using interval notation.

We need to find all the values of x that make the quotient $\frac{x+1}{-x-6}$ *positive*.

To find these values, we do a sign analysis.

We first look at the sign of the numerator $x+1$.

If $x=-1$, then $x+1$ is 0 .
If $x<-1$, then $x+1$ is negative.
If $x>-1$, then $x+1$ is positive.

We show this on the number line.

		-1	
sign of $x+1$:	− − − − − − − − − − − − − − − −	**0**	+ + + + + + +

We then look at the sign of the denominator $-x-6$.

If $x=-6$, then $-x-6$ is 0 .
If $x<-6$, then $-x-6$ is positive.
If $x>-6$, then $-x-6$ is negative.

		-6		-1	
sign of $x+1$:	− − − − − − −	−	− − − − − − − −	**0**	+ + + + + + +
sign of $-x-6$:	+ + + + + + +	**0**	− − − − − − − −	−	− − − − − − −

Finally, to find the sign of $\frac{x+1}{-x-6}$, we use the rules for dividing signed numbers.

		-6		-1	
sign of $x+1$:	− − − − − − −	−	− − − − − − − −	**0**	+ + + + + + +
sign of $-x-6$:	+ + + + + + +	**0**	− − − − − − − −	−	− − − − − − −
	negative quotient	undefined	positive quotient	0	negative quotient

So, we have $\frac{x+1}{-x-6} > 0$ when $-6 < x$ and $x < -1$.

Note that for $x=-6$, the quotient is undefined and so -6 *is not* part of the solution.
For $x=-1$, the quotient is 0 and so -1 *is not* part of the solution.
We write the solution in interval notation.

$$(-6, -1)$$

2c)

$$\frac{(u-4)(u+1)}{u-3} \le 0$$

Hint : Where are the only places where the rational expression might change signs?

Step 2
Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$u=-1 \qquad u=4$$

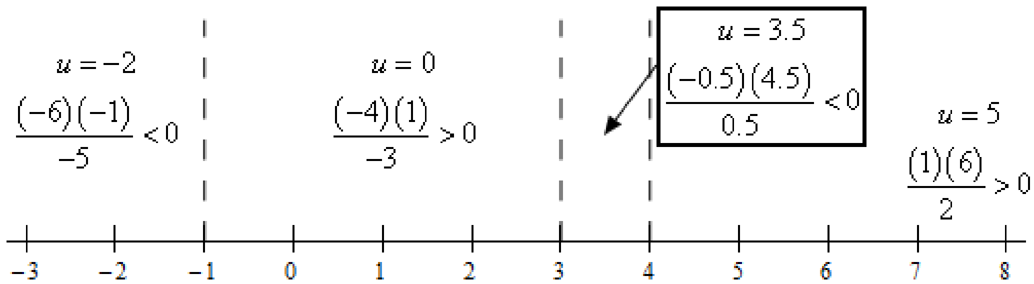
and the denominator will be zero at,

$$u=3$$

Hint : Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3
Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$u \le -1$	and	$3 < u \le 4$
$(-\infty,-1]$	and	$(3,4]$

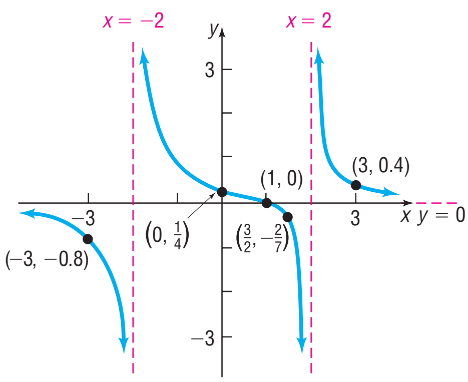
approach to help us understand the algebraic procedure for solving inequalities involving rational expressions.

EXAMPLE 3 Solving a Rational Inequality Using Its Graph

Solve $\frac{x-1}{x^2-4} \geq 0$ by graphing $R(x) = \frac{x-1}{x^2-4}$.

Solution Graph $R(x) = \frac{x-1}{x^2-4}$ and determine the intervals of x such that the graph is above or on the x-axis. Do you see why? These values of x result in $R(x)$ being positive or zero. We graphed $R(x) = \frac{x-1}{x^2-4}$ in Example 1, Section 5.3 (pp. 353–354). We reproduce the graph in Figure 47.

Figure 47



From the graph, we can see that $R(x) \geq 0$ for $-2 < x \leq 1$ or $x > 2$. The solution set is $\{x \mid -2 < x \leq 1 \text{ or } x > 2\}$ or, using interval notation, $(-2, 1] \cup (2, \infty)$.

Now Work PROBLEM 33

To solve a rational inequality algebraically, we follow the same approach that we used to solve a polynomial inequality algebraically. However, we must also identify the zeros of the denominator of the rational function, because the sign of a rational function may change on either side of a vertical asymptote. Convince yourself of this by looking at Figure 47. Notice that the function values are negative for $x < -2$ and are positive for $x > -2$ (but less than 1).

EXAMPLE 4 How to Solve a Rational Inequality Algebraically

Solve the inequality $\frac{4x+5}{x+2} \geq 3$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a rational expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$\frac{4x+5}{x+2} \geq 3$$

$$\frac{4x+5}{x+2} - 3 \geq 0 \qquad \text{Subtract 3 from both sides of the inequality.}$$

$$\frac{4x+5}{x+2} - 3 \cdot \frac{x+2}{x+2} \geq 0 \qquad \text{Multiply 3 by } \frac{x+2}{x+2}.$$

$$\frac{4x+5-3x-6}{x+2} \geq 0 \qquad \text{Write as a single quotient.}$$

$$\frac{x-1}{x+2} \geq 0 \qquad \text{Combine like terms.}$$

Step 2: Determine the real zeros (*x*-intercepts of the graph) of *f* and the real numbers for which *f* is undefined.

The zero of $f(x) = \dfrac{x-1}{x+2}$ is 1. Also, *f* is undefined for $x = -2$.

Step 3: Use the zeros and undefined values found in Step 2 to divide the real number line into intervals.

Use the zero and undefined value to separate the real number line into three intervals:

$$(-\infty, -2) \qquad (-2, 1) \qquad (1, \infty)$$

Step 4: Select a number in each interval, evaluate *f* at the number, and determine whether *f* is positive or negative. If *f* is positive, all values of *f* in the interval are positive. If *f* is negative, all values of *f* in the interval are negative.

Select a test number in each interval found in Step 3 and evaluate $f(x) = \dfrac{x-1}{x+2}$ at each number to determine if $f(x)$ is positive or negative. See Table 17.

Table 17

	-2	1	x
Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
Number chosen	-3	0	2
Value of *f*	$f(-3) = 4$	$f(0) = -\frac{1}{2}$	$f(2) = \frac{1}{4}$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is positive or zero, we conclude that $f(x) \geq 0$ for all numbers x for which $x < -2$ or $x \geq 1$. Notice we do not include -2 in the solution because -2 is not in the domain of f. The solution set of the inequality $\dfrac{4x+5}{x+2} \geq 3$ is $\{x \mid x < -2 \text{ or } x \geq 1\}$ or, using interval notation, $(-\infty, -2) \cup [1, \infty)$. Figure 48 shows the graph of the solution set.

Figure 48

Now Work PROBLEM 39

SUMMARY Steps for Solving Polynomial and Rational Inequalities Algebraically

STEP 1: Write the inequality so that a polynomial or rational expression *f* is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient and find the domain of *f*.

STEP 2: Determine the real numbers at which the expression *f* equals zero and, if the expression is rational, the real numbers at which the expression *f* is undefined.

STEP 3: Use the numbers found in Step 2 to separate the real number line into intervals.

STEP 4: Select a number in each interval and evaluate *f* at the number.

(a) If the value of *f* is positive, then $f(x) > 0$ for all numbers *x* in the interval.

(b) If the value of *f* is negative, then $f(x) < 0$ for all numbers *x* in the interval.

If the inequality is not strict ($\geq$ or $\leq$), include the solutions of $f(x) = 0$ that are in the domain of *f* in the solution set. Be careful to exclude values of *x* where *f* is undefined.

5.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve the inequality $3 - 4x > 5$. Graph the solution set. (pp. 123–125)

2. Solve the inequality $x^2 - 5x \leq 24$. Graph the solution set. (pp. 309–311)

Notice that the restriction $x = 1$ corresponds to a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set, the restriction is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

Example 5

2e) Solve: $\frac{7}{x+3} < 2$.

Solution:

Step 1: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$
$$\frac{7}{x+3} - 2 < 0$$

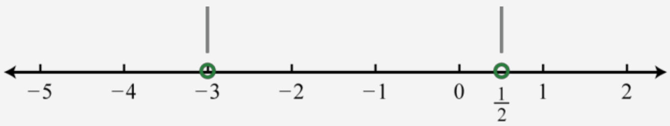
Step 2: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

$$\frac{7}{x+3} - \frac{2}{1} < 0$$
$$\frac{7 - 2(x+3)}{x+3} < 0$$
$$\frac{7 - 2x - 6}{x+3} < 0$$
$$\frac{-2x+1}{x+3} < 0$$

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

Root
$-2x+1=0$
$-2x=-1$
$x=\frac{1}{2}$
Restriction
$x+3=0$
$x=-3$

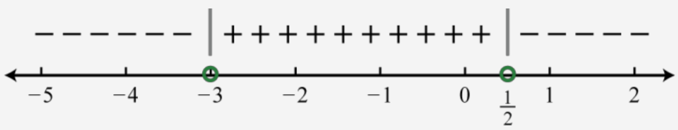
In this case, the strict inequality indicates that we should use an open dot for the root.



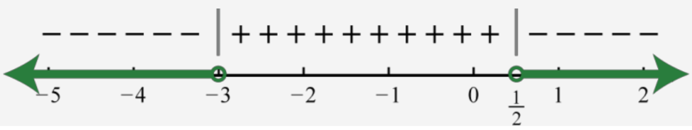
Step 3: Create a sign chart. Choose test values −4, 0, and 1.

$$f(-4)=\frac{-2(-4)+1}{-4+3}=\frac{+}{-}=-$$
$$f(0)=\frac{-2(0)+1}{0+3}=\frac{+}{+}=+$$
$$f(1)=\frac{-2(1)+1}{1+3}=\frac{-}{+}=-$$

And we have



Step 4: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x)<0$. Shade the appropriate values and then present your answer using interval notation.



28

$$\frac{3x+8}{x-1} < -2$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

So, we first need to add 2 to both sides to get,

$$\frac{3x+8}{x-1} + 2 < 0$$

We now need to combine the two terms in to a single rational expression.

$$\frac{3x+8}{x-1} + \frac{2(x-1)}{x-1} < 0$$
$$\frac{3x+8+2x-2}{x-1} < 0$$
$$\frac{5x+6}{x-1} < 0$$

At this point we can also see that factoring will not be needed for this problem.

Hint : Where are the only places where the rational expression might change signs?

Step 2
Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$x = -\frac{6}{5}$$

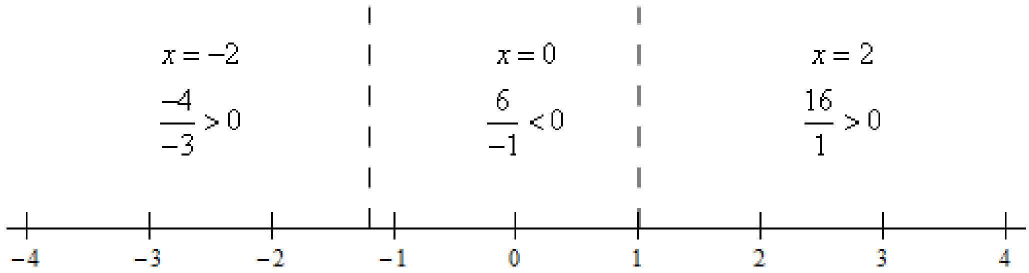
and the denominator will be zero at,

$$x = 1$$

Hint : Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3
Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$$\boxed{\begin{array}{c} -\frac{6}{5} < x < 1 \\ \left(-\frac{6}{5},1\right) \end{array}}$$

5. Solve the following inequality.

$$u \le \frac{4}{u-3}$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

So, we first need to get zero on one side of the inequality.

$$u - \frac{4}{u-3} \le 0$$

We now need to combine the two terms in to a single rational expression.

$$\begin{aligned} \frac{u\left(u-3\right)}{u-3} - \frac{4}{u-3} &\le 0 \\ \frac{u^2-3u-4}{u-3} &\le 0 \end{aligned}$$

Finally, we need to factor the numerator.

135 NOTES ON RATIONAL INEQUALITIES

Solving a rational inequality: Problem type 2

Solve the following inequality.

$$\frac{7-x}{x+1} > \frac{4-x}{x+3}$$

Write your answer as an interval or union of intervals.

If there is no real solution, click on "No solution".

We first rewrite the inequality so that 0 is on one side.

$$\frac{7-x}{x+1} - \frac{4-x}{x+3} > 0$$

Then, we find a common denominator and simplify.

$$\frac{(7-x)(x+3)-(4-x)(x+1)}{(x+1)(x+3)} > 0$$

$$\frac{7x+21-x^2-3x-\left(4x+4-x^2-x\right)}{(x+1)(x+3)} > 0$$

$$\frac{21+4x-x^2-\left(4+3x-x^2\right)}{(x+1)(x+3)} > 0$$

$$\frac{21+4x-x^2-4-3x+x^2}{(x+1)(x+3)} > 0$$

$$\frac{x+17}{(x+1)(x+3)} > 0$$

A rational expression can only change signs at the x-values that make its numerator or denominato

For the rational expression $\frac{x+17}{(x+1)(x+3)}$, we get the following.

Zero(s) of the numerator:

$$x+17 = 0$$
$$x = -17$$

Zero(s) of the denominator:

$$(x+1)(x+3) = 0$$
$$x+1 = 0 \quad \text{or} \quad x+3 = 0$$
$$x = -1 \qquad\quad x = -3$$

The values $x=-17$, $x=-3$, and $x=-1$ split the real numbers into the following intervals.

$(-\infty,-17)$, $(-17,-3)$, $(-3,-1)$, $(-1,\infty)$

For each interval, the rational expression will always be positive or always be negative.
So, we choose a test value in each interval and evaluate the rational expression at that value.
The sign of the rational expression at the test value tells us the sign for the entire interval.

Interval	**Test value** (x)	**Value of** $\frac{x+17}{(x+1)(x+3)}$
$(-\infty,-17)$	-18	$\frac{-18+17}{(-18+1)(-18+3)} < 0$
$(-17,-3)$	-4	$\frac{-4+17}{(-4+1)(-4+3)} > 0$
$(-3,-1)$	-2	$\frac{-2+17}{(-2+1)(-2+3)} < 0$
$(-1,\infty)$	0	$\frac{0+17}{(0+1)(0+3)} > 0$

We are trying to solve the inequality $\frac{x+17}{(x+1)(x+3)} > 0$.

So, the solution is the set of all x-values that make the rational expression ***positive***.

From the table above, we see that the rational expression is positive on the following intervals.

$(-17,-3)$
$(-1,\infty)$

Here is the answer.

$(-17,-3)\cup(-1,\infty)$

Be careful with the endpoints for this problem. Because we have an equal sign in the original inequality we need to include $u=-1$ and $u=4$ because the numerator and hence the rational expression will be zero there. However, we can't include $u=3$ because the denominator is zero there and so the rational expression has division by zero at that point!

---

6. Solve the following inequality.

2e)

$$\frac{t^3-6t^2}{t-2}>0$$

Step 1
The first thing we need to do is get a zero on one side of the inequality and then, if possible, factor the numerator and denominator as much as possible.

We already have zero on one side of the inequality but we still need to factor the numerator.

$$\frac{t^2\left(t-6\right)}{t-2}>0$$

Hint : Where are the only places where the rational expression might change signs?

Step 2
Recall from the discussion in the notes for this section that the rational expression can only change sign where the numerator is zero and/or where the denominator is zero.

We can see that the numerator will be zero at,

$$t=0 \qquad\qquad t=6$$

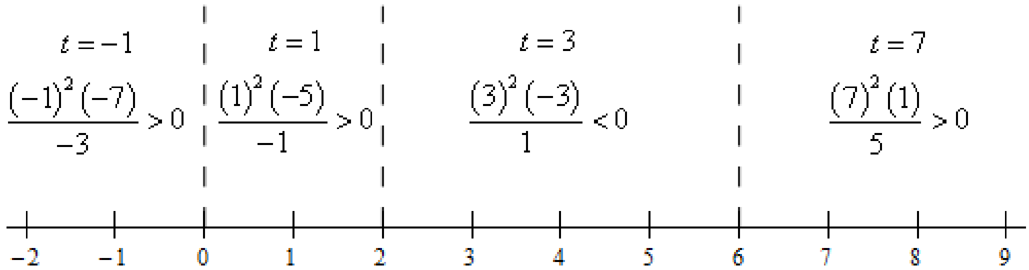
and the denominator will be zero at,

$$t=2$$

Hint : Knowing that the rational expression can only change sign at the points above how can we quickly determine if the rational expression is positive or negative in the ranges between those points?

Step 3
Just as we did with polynomial inequalities all we need to do is check the rational expression at test points in each region between the points from the previous step. The rational expression will have the same sign as the sign at the test point since it can only change sign at those points.

Here is a sketch of a number line with the points from the previous step graphed on it. We'll also show the test point computations on the number line as well. Here is the number line.



Step 4
All we need to do now is get the solution from the number line in the previous step. Here is both the inequality and interval notation from of the answer.

$t<0$	$0<t<2$	and	$t>6$
$(-\infty,0)$	$(0,2)$	and	$(6,\infty)$

Be careful to not include $t=0$ in the answer! It might be tempting to "simplify" the first two inequalities in our answer into a single inequality. However, we're looking for where the rational expression is positive only and at $t=0$ the rational expression is zero and so we need to exclude $t=0$ from our answer.

135 NOTES ON RATIONAL INEQUALITIES

2i)

Solving a rational inequality: Problem type 2

Solve the following inequality.

$$\frac{x-7}{x+5} \geq -3$$

Write your answer as an interval or union of intervals.

If there is no real solution, click on "No solution".

We first rewrite the inequality so that 0 is on one side.

$$\frac{x-7}{x+5}+3 \geq 0$$

We first rewrite the inequality so that 0 is on one side.

$$\frac{x-7}{x+5}+3 \geq 0$$

Then, we find a common denominator and simplify.

$$\frac{x-7+3(x+5)}{x+5} \geq 0$$

$$\frac{x-7+3x+15}{x+5} \geq 0$$

$$\frac{4x+8}{x+5} \geq 0$$

A rational expression can only change signs at the x-values that make its numerator or denominator zero.

For the rational expression $\frac{4x+8}{x+5}$, we get the following.

Zero(s) of the numerator:

$$4x+8 = 0$$
$$4x = -8$$
$$x = -2$$

Zero(s) of the denominator:

$$x+5 = 0$$
$$x = -5$$

The values $x=-5$ and $x=-2$ split the real numbers into the following intervals.

$$(-\infty,-5),\ (-5,-2),\ (-2,\infty)$$

For each interval, the rational expression will always be positive or always be negative.
So, we choose a test value in each interval and evaluate the rational expression at that value.
The sign of the rational expression at the test value tells us the sign for the entire interval.

Interval	Test value (x)	Value of $\frac{4x+8}{x+5}$
$(-\infty,-5)$	-6	$\frac{4(-6)+8}{-6+5} > 0$
$(-5,-2)$	-3	$\frac{4(-3)+8}{-3+5} < 0$
$(-2,\infty)$	0	$\frac{4(0)+8}{0+5} > 0$

We are trying to solve the inequality $\frac{4x+8}{x+5} \geq 0$.

So, the solution is the set of all x-values that make the rational expression ***positive or*** 0 .

From the table above, we see that the rational expression is positive on the following intervals.

$$(-\infty,-5)$$
$$(-2,\infty)$$

A rational expression equals 0 when its numerator equals 0 and its denominator does not.
So, our rational expression equals 0 at $x=-2$.

Including $x=-2$ with the intervals above gives the answer.

$$(-\infty,-5)\cup[-2,\infty)$$

Answer: $(-\infty,-2]\cup[-1,2]$

Try this! Solve: $-3x^4+12x^3-9x^2>0$.

Answer: $(1,3)$

Solving Rational Inequalities

A rational inequality is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function in the set of critical numbers.

Example 4

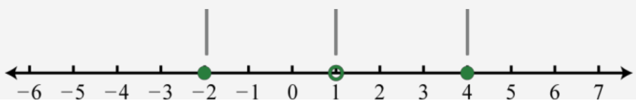
Solve: $\dfrac{(x-4)(x+2)}{(x-1)}\geq 0$.

Solution:

The zeros of a rational function occur when the numerator is zero and the values that produce zero in the denominator are the restrictions. In this case,

Roots (Numerator)	*Restriction (Denominator)*
$x-4$ or $x+2=0$	$x-1=0$
$x=4$ or $x=-2$	$x=1$

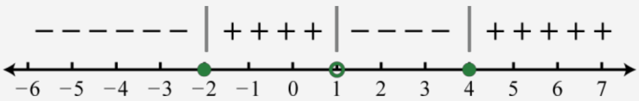
Therefore the critical numbers are −2, 1, and 4. Because of the inclusive inequality ($\geq$) use a closed dot for the roots {−2, 4} and always use an open dot for restrictions {1}. Restrictions are never included in the solution set.



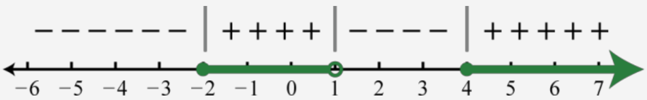
Use test values $x = -4, 0, 2, 6.$

$$f(-4) = (-4-4)(-4+2)(-4-1) = (-)(-)(-) = -$$
$$f(0) = (0-4)(0+2)(0-1) = (-)(+)(-) = +$$
$$f(2) = (2-4)(2+2)(2-1) = (-)(+)(+) = -$$
$$f(6) = (6-4)(6+2)(6-1) = (+)(+)(+) = +$$

And then complete the sign chart.

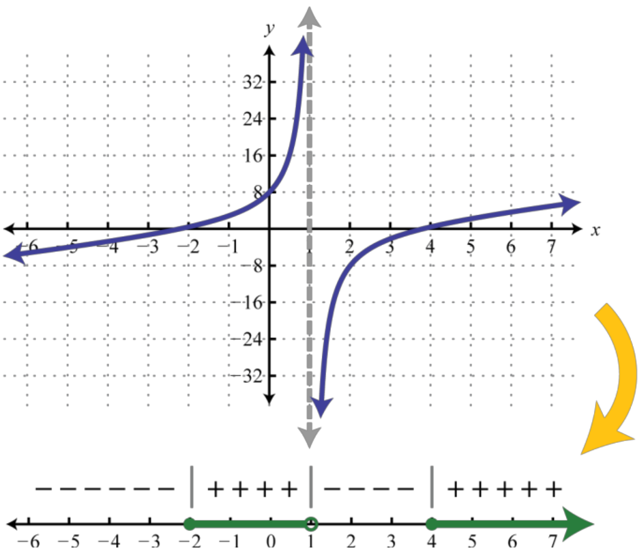


The question asks us to find the values for which $f(x) \geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.



Answer: $[-2,1) \cup [4, \infty)$

Graphing such rational functions like the one in the previous example is beyond the scope of this textbook. However, the graph of this function is provided below. Compare the graph to its corresponding sign chart.



Answer: $(-\infty,-3)\cup(\frac{1}{2},\infty)$

Example 6

Solve: $\frac{1}{x^2-4}\le\frac{1}{2-x}$.

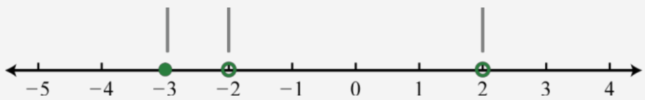
Solution:

Begin by obtaining zero on the right side.

$$\frac{1}{x^2-4}\le\frac{1}{2-x}$$
$$\frac{1}{x^2-4}-\frac{1}{2-x}\le 0$$

Next simplify the left side to a single algebraic fraction.

The critical numbers are −3, −2, and 2. Note that ±2 are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root −3.



Choose test values −4, $-2\frac{1}{2}=-\frac{5}{2}$, 0, and 3.

$$\frac{1}{x^2-4}-\frac{1}{2-x}\le 0$$

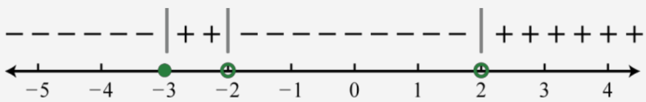
$$\frac{1}{(x+2)(x-2)}-\frac{1}{-(x-2)}\le 0$$

$$\frac{1}{(x+2)(x-2)}+\frac{1(x+2)}{(x-2)(x+2)}\le 0$$

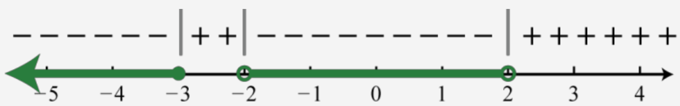
$$\frac{1+x+2}{(x+2)(x-2)}\le 0$$

$$\frac{x+3}{(x+2)(x-2)}\le 0$$

Construct a sign chart.



Answer the question; in this case, find x where $f(x)\le 0$.



Answer: $(-\infty,-3]\cup(-2,2)$

Try this! Solve: $\frac{2x^2}{2x^2+7x-4}\ge\frac{x}{x+4}$.

Answer: $(-4,0]\cup(\frac{1}{2},\infty)$

KEY CONCEPTS

- When a polynomial inequality is in standard form, with zero on one side, the roots of the polynomial are the critical numbers. Create a sign chart that models the function and then use it to answer the question.

$$\begin{aligned}\text{amount of work done together} &= \text{rate of working together} \cdot \text{time spent working together}\\ 1\,\text{garden} &= (\text{rate of working together}) \cdot (3\,\text{hours})\end{aligned}$$

From this, we find that the rate of Taylor and Carl working together is $\frac{1\,\text{garden}}{3\,\text{hours}} = \frac{1}{3}\frac{\text{garden}}{\text{hour}}$. We are asked to find out how long it would take for Carl to weed the garden on his own. Let us call this unknown t, measured in hours to be consistent with the other times given to us in the problem. Then:

$$\begin{aligned}\text{amount of work Carl does} &= \text{rate of Carl working} \cdot \text{time Carl spent working}\\ 1\,\text{garden} &= (\text{rate of Carl working}) \cdot (t\,\text{hours})\end{aligned}$$

In order to find t, we need to find the rate of Carl working, so let's call this quantity R, with units $\frac{\text{garden}}{\text{hour}}$. Using the fact that rates are additive, we have:

$$\begin{aligned}\text{rate working together} &= \text{rate of Taylor working} + \text{rate of Carl working}\\ \frac{1}{3}\frac{\text{garden}}{\text{hour}} &= \frac{1}{4}\frac{\text{garden}}{\text{hour}} + R\frac{\text{garden}}{\text{hour}}\end{aligned}$$

so that $R = \frac{1}{12}\frac{\text{garden}}{\text{hour}}$. Substituting this into our 'work-rate-time' equation for Carl, we get:

$$\begin{aligned}1\,\text{garden} &= (\text{rate of Carl working}) \cdot (t\,\text{hours})\\ 1\,\text{garden} &= \left(\frac{1}{12}\frac{\text{garden}}{\text{hour}}\right) \cdot (t\,\text{hours})\end{aligned}$$

Solving $1 = \frac{1}{12}t$, we get $t = 12$, so it takes Carl 12 hours to weed the garden on his own.[5] □

As is common with 'word problems' like Examples 4.3.2 and 4.3.3, there is no short-cut to the answer. We encourage the reader to carefully think through and apply the basic principles of rate to each (potentially different!) situation. It is time well spent. We also encourage the tracking of units, especially in the early stages of the problem. Not only does this promote uniformity in the units, it also serves as a quick means to check if an equation makes sense.[6]

Our next example deals with the average cost function, first introduced on page 82, as applied to PortaBoy Game systems from Example 2.1.5 in Section 2.1.

3) **Example 4.3.4.** Given a cost function $C(x)$, which returns the total cost of producing x items, recall that the average cost function, $\overline{C}(x) = \frac{C(x)}{x}$ computes the cost per item when x items are produced. Suppose the cost C, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function $\overline{C}(x)$.

2. Solve $\overline{C}(x) < 100$ and interpret.

[5]Carl would much rather spend his time writing open-source Mathematics texts than gardening anyway.
[6]In other words, make sure you don't try to add apples to oranges!

3. Determine the behavior of $\overline{C}(x)$ as $x \to \infty$ and interpret.

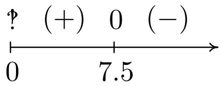
Solution.

1. From $\overline{C}(x) = \frac{C(x)}{x}$, we obtain $\overline{C}(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $\overline{C}(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.

2. Solving $\overline{C}(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned} \frac{80x+150}{x} &< 100 \\ \frac{80x+150}{x} - 100 &< 0 \\ \frac{80x+150-100x}{x} &< 0 \quad \text{common denominator} \\ \frac{150-20x}{x} &< 0 \end{aligned}$$

   If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.



   In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. When we apply Theorem 4.2 to $\overline{C}(x)$ we find that $y = 80$ is a horizontal asymptote to the graph of $y = \overline{C}(x)$. To more precisely determine the behavior of $\overline{C}(x)$ as $x \to \infty$, we first use long division[7] and rewrite $\overline{C}(x) = 80 + \frac{150}{x}$. As $x \to \infty$, $\frac{150}{x} \to 0^+$, which means $\overline{C}(x) \approx 80 + \text{very small } (+)$. Thus the average cost per system is getting closer to \$80 per system. If we set $\overline{C}(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 2.1.5, we realize \$80 is the variable cost per system –

[7] In this case, long division amounts to term-by-term division.

(24)

If we multiply an inequality, we need to consider two cases:
1) if we multiply by a positive expression, nothing changes,
2) if we multiply by a negative expression, the sign mark changes.

The solution is then combination of both possibilities.

3. Determine the behavior of $\overline{C}(x)$ as $x \to \infty$ and interpret.

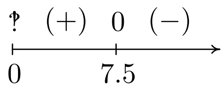
Solution.

1. From $\overline{C}(x) = \frac{C(x)}{x}$, we obtain $\overline{C}(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $\overline{C}(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.

2. Solving $\overline{C}(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned} \frac{80x+150}{x} &< 100 \\ \frac{80x+150}{x} - 100 &< 0 \\ \frac{80x+150-100x}{x} &< 0 \quad \text{common denominator} \\ \frac{150-20x}{x} &< 0 \end{aligned}$$

   If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.



   In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. When we apply Theorem 4.2 to $\overline{C}(x)$ we find that $y = 80$ is a horizontal asymptote to the graph of $y = \overline{C}(x)$. To more precisely determine the behavior of $\overline{C}(x)$ as $x \to \infty$, we first use long division[7] and rewrite $\overline{C}(x) = 80 + \frac{150}{x}$. As $x \to \infty$, $\frac{150}{x} \to 0^+$, which means $\overline{C}(x) \approx 80 + \text{very small } (+)$. Thus the average cost per system is getting closer to \$80 per system. If we set $\overline{C}(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 2.1.5, we realize \$80 is the variable cost per system –

[7] In this case, long division amounts to term-by-term division.

5.4 Polynomial and Rational Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Linear Inequalities (Section 1.5, pp. 123–125)
- Solving Quadratic Inequalities (Section 4.5, pp. 309–311)

Now Work the **'Are You Prepared?'** problems on page 371.

OBJECTIVES
1 Solve Polynomial Inequalities (p. 368)
2 Solve Rational Inequalities (p. 369)

1 Solve Polynomial Inequalities

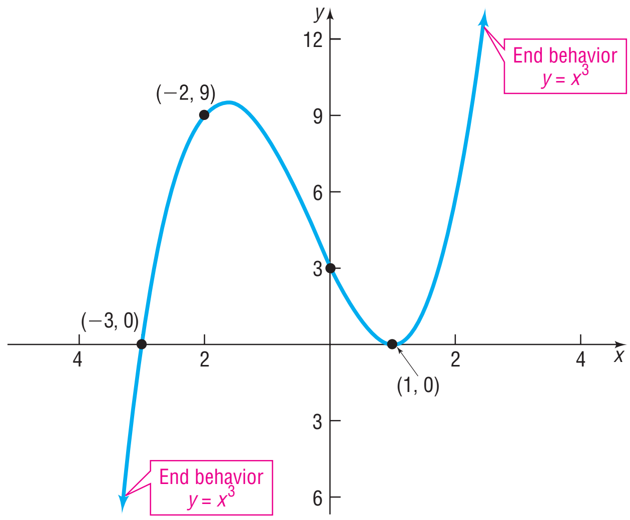
In this section we solve inequalities that involve polynomials of degree 3 and higher, along with inequalities that involve rational functions. To help understand the algebraic procedure for solving such inequalities, we use the information obtained in the previous three sections about the graphs of polynomial and rational functions. The approach follows the same methodology that we used to solve inequalities involving quadratic functions.

EXAMPLE 1 Solving a Polynomial Inequality Using Its Graph

Solve $(x + 3)(x - 1)^2 > 0$ by graphing $f(x) = (x + 3)(x - 1)^2$.

Solution Graph $f(x) = (x + 3)(x - 1)^2$ and determine the intervals of x for which the graph is above the x-axis. These values of x result in $f(x)$ being positive. Using Steps 1 through 6 on page 333, we obtain the graph shown in Figure 45.

Figure 45



From the graph, we can see that $f(x) > 0$ for $-3 < x < 1$ or $x > 1$. The solution set is $\{x \mid -3 < x < 1 \text{ or } x > 1\}$ or, using interval notation, $(-3, 1) \cup (1, \infty)$.

Now Work PROBLEM 9

The results of Example 1 lead to the following approach to solving polynomial and rational inequalities algebraically. Suppose that the polynomial or rational inequality is in one of the forms

$$f(x) < 0 \qquad f(x) > 0 \qquad f(x) \le 0 \qquad f(x) \ge 0$$

approach to help us understand the algebraic procedure for solving inequalities involving rational expressions.

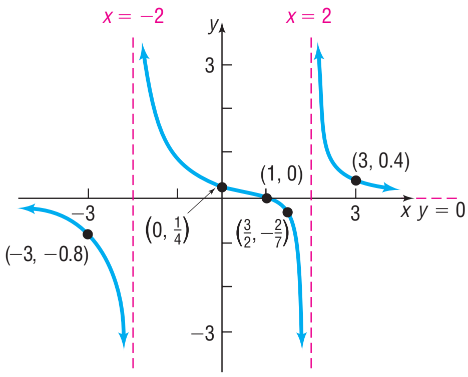
EXAMPLE 3 Solving a Rational Inequality Using Its Graph

35)

Solve $\dfrac{x-1}{x^2-4} \geq 0$ by graphing $R(x) = \dfrac{x-1}{x^2-4}$.

Solution Graph $R(x) = \dfrac{x-1}{x^2-4}$ and determine the intervals of x such that the graph is above or on the x-axis. Do you see why? These values of x result in $R(x)$ being positive or zero. We graphed $R(x) = \dfrac{x-1}{x^2-4}$ in Example 1, Section 5.3 (pp. 353–354). We reproduce the graph in Figure 47.

Figure 47



From the graph, we can see that $R(x) \geq 0$ for $-2 < x \leq 1$ or $x > 2$. The solution set is $\{x \mid -2 < x \leq 1 \text{ or } x > 2\}$ or, using interval notation, $(-2, 1] \cup (2, \infty)$.

Now Work PROBLEM 33

To solve a rational inequality algebraically, we follow the same approach that we used to solve a polynomial inequality algebraically. However, we must also identify the zeros of the denominator of the rational function, because the sign of a rational function may change on either side of a vertical asymptote. Convince yourself of this by looking at Figure 47. Notice that the function values are negative for $x < -2$ and are positive for $x > -2$ (but less than 1).

EXAMPLE 4 How to Solve a Rational Inequality Algebraically

Solve the inequality $\dfrac{4x+5}{x+2} \geq 3$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a rational expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$\frac{4x+5}{x+2} \geq 3$$

$$\frac{4x+5}{x+2} - 3 \geq 0 \quad \text{Subtract 3 from both sides of the inequality.}$$

$$\frac{4x+5}{x+2} - 3 \cdot \frac{x+2}{x+2} \geq 0 \quad \text{Multiply 3 by } \frac{x+2}{x+2}.$$

$$\frac{4x+5-3x-6}{x+2} \geq 0 \quad \text{Write as a single quotient.}$$

$$\frac{x-1}{x+2} \geq 0 \quad \text{Combine like terms.}$$

(6a) $x \in (1, 3]$ $\rightarrow x \neq 1$

$$\frac{3-x}{x-1} \geq 0$$

(6b) $x \leq -1$ $x \geq 0$ $x < 1$

$$\frac{x\,(-x - 1)}{(1-x)} \geq 0$$