PROBLEM E1

1pt

$$\left(\begin{array}{cccc|cccc} 5 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \overset{-5,\,-1}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\overset{+5,\,-4}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & -5 & -5 & -4 & 1 & 0 & -5 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 20 & -4 & 1 & 5 & -5 & 0 \\ 0 & 0 & -16 & 0 & 0 & -4 & -1 & 1 \end{array}\right) \overset{+4/5}{\sim}$$

5pt

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 20 & -4 & 1 & 5 & -5 & 0 \\ 0 & 0 & 0 & -\frac{16}{5} & \frac{1}{5} & 0 & -5 & 1 \end{array}\right) \overset{-1,\,+1/5}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{20} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{25}{16} & -\frac{5}{16} \end{array}\right)$$

$$\overset{-1,\,5}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & \cancel{0}\frac{1}{4} & 0 & -\frac{9}{16} & \cancel{0}\frac{5}{16} \\ 0 & 1 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{16} & -\frac{1}{16} \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{25}{16} & -\frac{5}{16} \end{array}\right) \overset{-1}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{5}{8} & \frac{3}{8} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{5}{16} & \frac{5}{16} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{16} & -\frac{1}{16} \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{25}{16} & -\frac{5}{16} \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & 0 & -5/16 & 1/16 \\ 0 & 1 & 0 & 0 & 0 & -1/4 & -5/16 & 5/16 \\ 0 & 0 & 1 & 0 & 0 & 1/4 & 1/16 & -1/16 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{25}{16} & -\frac{5}{16} \end{array}\right)$$

A^{-1} 1pt

$$B^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{5}{4} & 0 & \frac{1}{4} \\ \frac{5}{4} & -\frac{5}{4} & -\frac{1}{4} & 0 \\ -1/4 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{5}{4} & \frac{25}{4} & 0 & -\frac{1}{4} \end{pmatrix}$$

3pts

PROBLEME 2

(B)

$f(x,y) = \arcsin(x^2+y^2-2)$

D_f: $D_{\arcsin} = \langle -1,1 \rangle$

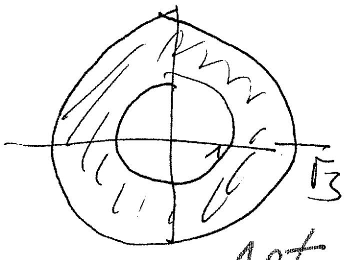
$-1 \le x^2+y^2-2 \le 1$

$1 \le x^2+y^2 \le 3$

2pts

$D_f = \{[x,y] \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 3\}$

including boundary



1pt

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(x^2+y^2-2)^2}} \cdot 2x \quad 1pt$$

for $1 < x^2+y^2 < 3$ 1pt

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(x^2+y^2-2)^2}} \cdot 2y \quad 1pt$$

remaining points:

$x^2+y^2=3$ .. no need to compute, no sense 1pt

$x^2+y^2=1$: only $\frac{\partial f}{\partial x}(0,1), \frac{\partial f}{\partial x}(0,-1), \frac{\partial f}{\partial y}(1,0), \frac{\partial f}{\partial y}(1,0)$ 1pt

1pt

$$\frac{\partial f}{\partial x}(0,1) = \lim_{h\to 0} \frac{\arcsin(\cancel{h}\, h^2-1) - \arcsin(-1)}{h} =$$

$$\overset{l'H}{\underset{\frac{0}{0}}{=}} \lim_{h\to 0} \frac{1}{\sqrt{1-(h^2-1)^2}} \cdot 2h = \lim_{h\to 0} \frac{1}{\sqrt{-h^4+2h^2}} \cdot 2h =$$

$$= \lim_{h\to 0} \frac{1}{|h|\sqrt{2-h^2}} \cdot 2h \quad \text{... does not exist}$$

$\frac{\partial f}{\partial x}(0,-1) = \lim_{h\to 0} \frac{\arcsin(h^2-1)-\arcsin(-1)}{h}$ ... the same

1pt $\frac{\partial f}{\partial y}(1,0) = \lim_{h\to 0} \arcsin\frac{(h^2-1)}{2}$ ... the same $\frac{\partial f}{\partial y}(-1,0)$ the same

(:a) $\boxed{\text{PROBLEM E3}}$ 4

$$\underbrace{4\,\mathrm{arctg}(xy) + \sin(y+x) + \pi}_{F(x,y)} = 0 \quad , \quad [-1,1]$$

3pts
① $F \in C^\infty(\mathbb{R}^2)$

② $F(-1,1) = 4 \cdot \mathrm{arctg}(-1) + \sin(1-1) + \pi = 0$

③ $\frac{\partial F}{\partial y}(-1,1) = \left(4 \cdot \frac{1}{1+x^2y^2} \cdot x + \cos(y+x) \cdot 1\right)_{\substack{x=-1 \\ y=1}} = 4 \cdot \frac{1}{1+1} \cdot (-1) + \cos 0$
$= -2 + 1 = -1 \neq 0$

$\Rightarrow$ there exists C^∞ function f with required properties

1pt In part. $4\,\mathrm{arctg}(x f(x)) + \sin(f(x)+x) + \pi = 0$ in a nbhood of -1

differentiate:
1pt $\frac{4}{1+x^2(f(x))^2} \cdot (f(x) + x f'(x)) + \cos(f(x)+x)(f'(x)+1) = 0$

$x=-1,\ f(-1)=1$: $\frac{4}{1+1}(1 - f'(-1)) + \cos(1-1)(f'(-1)+1) = 0$

$2 - 2f'(-1) + f'(-1) + 1 = 0$

$f'(-1) = 3$

1pt

tangent line: $y = 1 + 3(x+1)$ 1pt

second derivative:
2pts
$- \frac{4}{(1+x^2(f(x))^2)^2} \cdot (2x(f(x))^2 + x^2 \cdot 2f(x)f'(x))(f(x)+xf'(x))$
$+ \frac{4}{1+x^2(f(x))^2}(f'(x) + f'(x) + xf''(x)) - \sin(f(x)+x)(f'(x)+1)^2 + \cos(f(x)+x) \cdot f''(x) = 0$

$x=-1,\ f(-1)=1,\ f'(-1)=3$:

$-\frac{4}{(1+1)^2}(-2 + 2 \cdot 3)(1-3) + \frac{4}{1+1}(3+3-f''(-1)) - \sin 0 \cdot (3+1)^2$
$+ \cos 0 \cdot f''(-1) = 0$

$-1 \cdot 4 \cdot (-2) + 2 \cdot 6 - 2f''(-1) + f''(-1) = 0$

$f''(-1) = 20$

1pt

PROBLEME 5

$f(x,y,z) = x - yz \qquad M = \{[x,y,z] \in \mathbb{R}^3 :\ x^2+3y^2+z^2=4,\ z+y\sqrt{3}+1 \le 0\}$

) Existence of extrema: • f is cts on $\mathbb{R}^3$

• M is closed ("=" and "≤") and bounded

$(x^2+3y^2+z^2=4 \Rightarrow (x,y,z) \in \overline{B(0,2)})$

2pts

$\Rightarrow M$ is compact

Thus extrema do exist provided $M \neq \emptyset$

② For future use:

$g_1(x,y,z) = x^2+3y^2+z^2-4 \ ,\ g_2(x,y,z) = z+y\sqrt{3}+1$

$Df = [1,-z,-y] \ ,\ Dg_1 = [2x, 6y, 2z] \ ,\ Dg_2 = [0,\sqrt{3},1]$

$M = M_1 \cup M_2$, where $M_1 = \{[x,y,z] \in \mathbb{R}^3 : x^2+3y^2+z^2=4,\ z+y\sqrt{3}+1 < 0\}$

1pt $\qquad M_2 = \{[x,y,z] \in \mathbb{R}^3 : x^2+3y^2+z^2=4,\ z+y\sqrt{3}+1 = 0\}$

On M_1: At points of extrema either $Dg_1 = 0$ (only at $[0,0,0] \notin M_1$, so this is not possible on M_1) 1pt

or $\exists \lambda : Df + \lambda Dg_1 = 0$

1pt
$$1 + \lambda \cdot 2x = 0$$
$$-z + \lambda \cdot 6y = 0 \quad \Rightarrow \quad 3y + xz = 0 \quad \longrightarrow \quad 3y - x^2 y = 0$$
$$-\cancel{y} + \lambda \cdot 2z = 0 \quad \Rightarrow \quad z + xy = 0 \Rightarrow z = -xy \qquad y(3-x^2) = 0 \Rightarrow y = 0 \text{ or } x^2 = 3$$
$$x^2+3y^2+z^2 = 4$$

$y = 0 \Rightarrow z = -xy = 0 \Rightarrow x^2 = 4 \Rightarrow [2,0,0]$ and $[-2,0,0]$

$z+y\sqrt{3}+1 = 1 > 0 \Rightarrow$ not in M_1

---

so, $x^2 = 3$, $x = \pm\sqrt{3}$, $z = -xy = \mp\sqrt{3}y \Rightarrow z^2 = 3y^2$

$3 + 3y^2 + 3y^2 = 4 \Rightarrow y^2 = \frac{1}{6} \Rightarrow y = \pm\frac{1}{\sqrt{6}}$

$[\sqrt{3}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}]$

$[\sqrt{3}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}]$

$[-\sqrt{3}, \frac{1}{\sqrt{6}}, +\frac{1}{\sqrt{2}}]$

$[-\sqrt{3}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}]$

2pts

$$z + y\sqrt{3} + 1 = \begin{cases} 1 \\ 1 \\ \sqrt{2}+1 \end{cases} > 0 \qquad -\sqrt{2}+1 < 0$$

$\Rightarrow$ only $[-\sqrt{3}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}] \in M_1$

$$f(\quad) = -\sqrt{3} - \frac{1}{2\sqrt{3}} = -\sqrt{3} - \frac{1}{6}\sqrt{3} = -\frac{7}{6}\sqrt{3}$$

On M_2: Case 1 Dg_1 and Dg_2 lin. dependent.

$Dg_1 = \alpha \cdot Dg_2$;

$2x = \alpha \cdot 0$

$6y = \alpha \cdot \sqrt{3}$

$2z = \alpha \cdot 1$

$\Rightarrow x = 0$

$6y = 2z \cdot \sqrt{3}$

$\Rightarrow z = \sqrt{3}y$

$z + \sqrt{3}y ~~z~~ + 1 = 0$

$2z + 1 = 0$

$z = -\frac{1}{2}$

$y = -\frac{\sqrt{3}}{2}$

2pts

$[0, -\frac{\sqrt{3}}{2}, \frac{1}{2}] \notin M$

$0^2 + 3 \cdot \frac{3}{4} + \frac{1}{4} = \frac{10}{4} \neq 4$

$\Rightarrow$ impossible

$Dg_2 = \alpha \cdot Dg_1 \Rightarrow$ necessary $\alpha \neq 0$ (since $Dg_2 \neq 0$) $\Rightarrow Dg_1 = \frac{1}{\alpha} Dg_2$, impossible by the above

Case 2: $\exists \lambda, \mu$: $Df + \lambda Dg_1 + \mu Dg_2 = 0$

~~$1 + \lambda Dg$~~

$1 + \lambda \cdot 2x + \mu \cdot 0 = 0$

$-z + \lambda \cdot 6y + \mu \cdot \sqrt{3} = 0$

$-y + \lambda \cdot 2z + \mu \cdot 1 = 0$

$x^2 + 3y^2 + z^2 = 4$

$~~z~~ + y\sqrt{3} + 1 = 0$

1pts

$-z + y\sqrt{3} + 2\lambda(3y - \sqrt{3}z) = 0$

$(2\sqrt{3}\lambda + 1)(y\sqrt{3} - z) = 0$

$z = y\sqrt{3}$ $\qquad$ $\lambda = -\frac{1}{2\sqrt{3}}$

$z = y\sqrt{3}$: $2z + 1 = 0 \Rightarrow z = -\frac{1}{2}$

$y = -\frac{\sqrt{3}}{2}$, $x^2 + 3 \cdot \frac{3}{4} + \frac{1}{4} = 4$

$x^2 + \frac{5}{2} = 4$

$\Rightarrow x^2 = \frac{3}{2}$

$[\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}]$

$[-\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}]$

$f\left(\sqrt{\frac{3}{2}}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \sqrt{\frac{3}{2}} - \frac{\sqrt{3}}{4} = \sqrt{3}\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)$

$f\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = -\sqrt{\frac{3}{2}} - \frac{\sqrt{3}}{4} = -\sqrt{3}\left(\frac{1}{\sqrt{2}} + \frac{1}{4}\right)$

$\lambda = -\frac{1}{2\sqrt{3}}$: $\quad 1 + \lambda \cdot 2x = 0 \Rightarrow 1 - \frac{x}{\sqrt{3}} = 0 \Rightarrow x = \sqrt{3}$

$\Downarrow$

$3y^2 + z^2 = 1$

$z + y\sqrt{3} + 1 = 0 \Rightarrow z = \cancel{0} - 1 - y\sqrt{3}$

$z^2 = 1 + 2\sqrt{3}y + 3y^2$

$\Rightarrow 3y^2 + 1 + 2\sqrt{3}y + 3y^2 = 1$

$6y^2 + 2y\sqrt{3} = 0$

3 pts

$y = 0$

$\cancel{z = 1}\ z = -1$

$\left.\begin{matrix} \cancel{[\sqrt{3}, 0, 1]} \\ [\cancel{0}\sqrt{3}, 0, -1] \end{matrix}\right\} f(\ ) = \sqrt{3}$

$6y + 2\sqrt{3} = 0$

$y = -\frac{1}{\sqrt{3}}$

$z^2 = 0$

$\left[\sqrt{3}, -\frac{1}{\sqrt{3}}, 0\right]$

$f(\ ) = \sqrt{3}$

Comparison: Maximum is $\sqrt{3}$ at $[\sqrt{3}, 0, -1]$ and $\left[\sqrt{3}, -\frac{1}{\sqrt{3}}, 0\right]$

$\left(\sqrt{3} > \sqrt{3}\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right)\right.$ since $\frac{1}{\sqrt{2}} - \frac{1}{4} \in (0,1)$, other values are negative)

<u>Minimum:</u>

$\left.\begin{matrix} \frac{7}{6} \overset{?}{>} \frac{1}{\sqrt{2}} + \frac{1}{4} \\ \frac{11}{12} = \frac{7}{6} - \frac{1}{4} > \frac{1}{\sqrt{2}} \\ \frac{121}{144} > \frac{1}{2} \text{ YES} \end{matrix}\right\} \Rightarrow$ Minimum is $-\frac{7}{6}\sqrt{3}$ at $\left[-\sqrt{3}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}\right]$

2 pts

PROBLEM E5

$$\int \frac{x^4+24}{(x+2)(x^3-8)}\,dx$$

① division: $(x+2)(x^3-8) = x^4+2x^3-8x-16$

$(x^4+24) : (x^4+2x^3-8x-16) = 1$ 1pt

$-(x^4+2x^3-8x-16)$

$-2x^3+8x+40$

$$\frac{x^4+24}{(x+2)(x^3-8)} = 1 + \frac{-2x^3+8x+40}{(x+2)(x^3-8)}$$

② partial fractions:

$$\frac{-2x^3+8x+40}{(x+2)(x-2)(x^2+2x+4)} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+2x+4}$$ 1pt

$-2x^3+8x+40 = A(x-2)(x^2+2x+4) + B(x+2)(x^2+2x+4) + C(x+D)(x^2-4)$

$\underline{x=2:}$ $-16+16+40 = B\cdot 4(4+4+4) \Rightarrow 40 = B\cdot 4\cdot 12 \Rightarrow$
$B = \frac{5}{6}$

$\underline{x=-2:}$ $16-16+40 = A\cdot(-4)(4-4+4) \Rightarrow 40 = A\cdot(-16)$
$A = -\frac{5}{2}$

at x^3: $-2 = +A + B + C$

$\Rightarrow C = -A - B - 2 = \frac{5}{2} - \frac{5}{6} - 2 = -\frac{1}{3}$ 3pts

at x^0: $40 = -8A + 8B - 4D$

$\Rightarrow D = -2A + 2B - 10 = 5 + \frac{5}{3} - 10 = -\frac{10}{3}$

$$\int \frac{-\frac{1}{3}x - \frac{10}{3}}{x^2+2x+4}\,dx = \int\left(-\frac{1}{6}\,\frac{2x+2}{x^2+2x+4} + \frac{-3}{x^2+2x+4}\right)dx$$ 1pt

$$= -\frac{1}{6}\lg(x^2+2x+4) + \int \frac{-3}{x^2+2x+4}\,dx$$ 1pt

$$\int -\frac{3}{x^2+2x+4}\,dx = -3\int \frac{1}{(x+1)^2+3}\,dx = -\int \frac{1}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1} =$$

$$= -\int \frac{\sqrt{3}\,\frac{1}{\sqrt{3}}}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}\,dx \overset{c}{=} -\sqrt{3}\,\operatorname{arctg}\frac{x+1}{\sqrt{3}} \quad \text{a } \mathbb{R}$$

3pts

Hence;

$$\int \frac{x^4+24}{(x+2)(x^3-8)}\,dx \overset{c}{=} x - \frac{5}{2}\overset{1pt}{\lg|x+2|} + \frac{5}{6}\overset{1pt}{\lg|x-2|} - \frac{1}{6}\lg(x^2+2x+4) - \sqrt{3}\cdot\operatorname{arctg}\frac{x+1}{\sqrt{3}}.$$

1pt

on each of the intervals

$(-\infty,-2), (-2,2), (2,+\infty)$. 2pts