MATRIX3

PROBLEM B1

$$\det\begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 2 & 3 & 5 & 9 & 17 \\ 3 & 5 & 8 & 13 & 20 \\ 2 & 4 & 8 & 8 & 16 \\ 16 & 8 & 4 & 2 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & -1 & -3 & -7 & -15 \\ 0 & -1 & -4 & -11 & -28 \\ 0 & 0 & 0 & -8 & -16 \\ 0 & -24 & -60 & -126 & -255 \end{pmatrix}$$

$$2 - 8\cdot 16 = 2(1-8\cdot 8) = 2\cdot(-63) = -126$$

$$= \det\begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & -1 & -3 & -7 & -15 \\ 0 & 0 & -1 & -4 & -13 \\ 0 & 0 & 0 & -8 & -16 \\ 0 & 0 & 12 & 42 & 105 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & -1 & -3 & -7 & -15 \\ 0 & 0 & -1 & -4 & -13 \\ 0 & 0 & 0 & -8 & -16 \\ 0 & 0 & 0 & -6 & -51 \end{pmatrix} =$$

$$-60 + 3\cdot 24 = -60 + 72 = 12$$

$$-126 + 7\cdot 24 = -126 + 168 = 42$$

$$-255 + 15\cdot 24 = 5(-51 + 3\cdot 24) = 5\cdot(-51+72) = 5\cdot 21 = 105$$

$$105 - 12\cdot 13 = = 105 - 144 - 12 = 93 - 144 = -51$$

7pts

$$= 1\cdot(-1)^{1+1}\det\begin{pmatrix} -1 & -3 & -7 & -15 \\ 0 & -1 & -4 & -13 \\ 0 & 0 & -8 & -16 \\ 0 & 0 & -6 & -51 \end{pmatrix} = (-1)\cdot(-1)^{1+1}\cdot\det\begin{pmatrix} -1 & -4 & -13 \\ 0 & -8 & -16 \\ 0 & -6 & -51 \end{pmatrix}$$

$$= -(-1)\cdot(-1)^{1+1}\det\begin{pmatrix} -8 & -16 \\ -6 & -51 \end{pmatrix} = 8\cdot 51 - 6\cdot 16 =$$

$$= 8(51 - 6\cdot 2) = 8\cdot(51-12) = 8\cdot 39 = 240 + 72 = 312$$

$$\det B = \left(-\tfrac{1}{2}\right)^5\cdot\det A = -\frac{312}{32} = -\frac{39}{4}$$

5 pue

PDY

PROBLEM B2

$$f(x,y) = (x^2+y^2-4)^{\sqrt{xy}} = \exp\left(\sqrt{xy}\cdot \lg(x^2+y^2-4)\right)$$

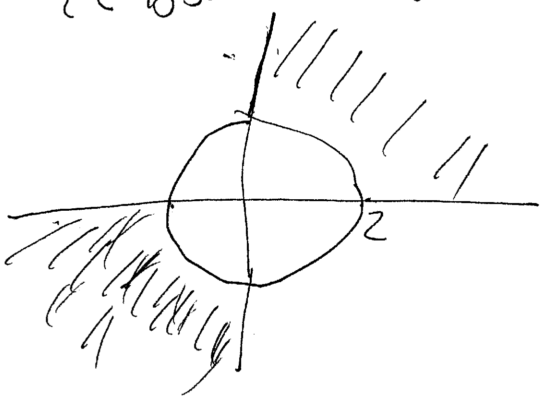
2pts

D_f: $x^2+y^2-4>0$ $\quad xy \geq 0$ .. $x \geq 0$ & $y \geq 0$

$x^2+y^2>4$ $\quad$ or $x \leq 0$ & $y \leq 0$

$D_f = \{[x,y] \in \mathbb{R}^2 : x^2+y^2>4$ & $(x \geq 0$ & $y \geq 0$ or $x \leq 0$ & $y \leq 0)$

1pt



with half-lines

without arcs

2pts

$$\frac{\partial f}{\partial x} = \exp\left(\sqrt{xy}\ \lg(x^2+y^2-4)\right)\cdot\left(\frac{1}{2\sqrt{xy}}\cdot y\ \lg(x^2+y^2-4) + \sqrt{xy}\ \frac{1}{x^2+y^2-4}\cdot 2x\right)$$

$$\frac{\partial f}{\partial y} = \exp\left(\sqrt{xy}\ \lg(x^2+y^2-4)\right)\left(\frac{1}{2\sqrt{xy}}x\ \lg(x^2+y^2-4) + \sqrt{xy}\ \frac{1}{x^2+y^2-4}\cdot 2y\right)$$

both for $[x,y] \in D_f$, $x \neq 0$, $y \neq 0$ $\quad$ 1pt

For $x=0$ or $y=0$:

1pt only $\frac{\partial f}{\partial x}(x,0) = 0$, because $f(x,0) = 1$ $\left(\forall x \in (-\infty,-2) \cup (2,\infty)\right)$

1pt and $\frac{\partial f}{\partial y}(0,y) = 0$, because $f(0,y) = 1$ $\left(y \in (-\infty,-2) \cup (2,\infty)\right)$

2pts

(otherwise no segment in domain:

at $[x,0]$ no vertical segment centered at $[x,0]$ contained in D_f

at $[0,y]$ no horizontal segment centered at $[0,y]$ contained in D_f

$\Rightarrow$ No need to compute $\frac{\partial f}{\partial y}(x,0)$ and $\frac{\partial f}{\partial x}(0,y)$.

IFT2!

PROBLEM B3

1

$$\underbrace{e^{x+y^2} - \cos(x+y)}_{F(x,y)} = 0 \quad , \quad [-1,1]$$

① $F \in C^\infty(\mathbb{R}^2)$

② $F(-1,1) = e^0 - \cos 0 = 0$

③ $\frac{\partial F}{\partial y}(-1,1) = \left(e^{x+y^2}\cdot 2y + \sin(x+y)\right)_{\substack{x=-1\\y=1}} = e^0 \cdot 2 + 0 = 2 \neq 0$

3pts

$\Rightarrow$ there exists a C^∞ function f with the required properties,

In part.: $e^{x+f(x)^2} - \cos(x+f(x)) = 0$ on a nbhood of -1 1pt

differentiate: $e^{x+f(x)^2}(1+2f(x)f'(x)) + \sin(x+f(x))(1+f'(x)) = 0$ 1pt

$x=-1, f(-1)=1$: $e^0(1+2f'(-1)) + \sin 0 \cdot (1+f'(-1)) = 0$

$f'(-1) = -\frac{1}{2}$ 1pt

tangent line: $y = 1 - \frac{1}{2}(x+1)$ 1pt

second derivative:

$e^{x+f(x)^2}(1+2f(x)f'(x))^2 + e^{x+f(x)^2}(2f'(x)\cdot f'(x) + 2f(x)f''(x))$

$+ \cos(x+f(x))(1+f'(x))^2 + \sin(x+f(x))\cdot f''(x) = 0$ 2pts

$x=-1, f(-1)=1, f'(-1) = -1/2$

$e^0 \cdot (1 + 2\cdot 1\cdot(-\frac{1}{2}))^2 + e^0\left(2\cdot(-\frac{1}{2})^2 + 2\cdot 1\cdot f''(-1)\right)$

$+ \cos 0\,(1-\frac{1}{2})^2 + \sin 0 \cdot f''(-1) = 0$

$0 + \frac{1}{2} + 2f''(-1) + \frac{1}{4} = 0$

$f''(-1) = -\frac{3}{8}$ 1pt

PROBLEM B4

$f(x,y,z) = xy \quad , \quad M = \{ [x,y,z] \in \mathbb{R}^3 : x^2+y^2+z^2=23, \ y+z \geq 4 \}$

- Existence of extrema:

2pts

f is cts on $\mathbb{R}^3$

M is closed ("=" and "≥" for cts functions)

and bdd (contained in the closed ball centered at 0 with radius $\sqrt{23}$)

So M is compact. Thus extrema do exist provided $M \neq \emptyset$.

- $M = M_1 \cup M_2$

1pt

$M_1 = \{ [x,y,z] \in \mathbb{R}^3 : x^2+y^2+z^2=23, \ y+z > 4 \}$

$M_2 = \{ [x,y,z] \in \mathbb{R}^3 : x^2+y^2+z^2=23, \ y+z = 4 \}$

For future use: $g_1(x,y,z) = x^2+y^2+z^2-23$

$g_2(x,y,z) = y+z-4$

$Df = [y,x,0] \ , \ Dg_1 = [2x,2y,2z] \ , \ Dg_2 = [0,1,1]$

- Extrema on M_1:

1pt Case 1: $Dg_1 = 0$ .. only at $[0,0,0] \notin M_1$ -- no points here

Case 2: $\exists \lambda \in \mathbb{R}$: $Df + \lambda Dg_1 = 0$

1pt
$$\begin{cases} y + \lambda \cdot 2x = 0 \\ x + \lambda \cdot 2y = 0 \\ 0 + \lambda \cdot 2z = 0 \end{cases}$$

2pts

$\lambda = 0 \Rightarrow y = x = 0 \Rightarrow z^2 = 23$

$z = \pm\sqrt{23}$

$[0,0,-\sqrt{23}] \notin M_1$

$[0,0,\sqrt{23}] \in M_1$, $f(0,0,\sqrt{23}) = 0$

$z = 0$: from the first two equations

$y^2 - x^2 = 0 \Rightarrow y = \pm x$

$2x^2 = 23$

$x^2 = \frac{23}{2} \Rightarrow x = \pm\sqrt{\frac{23}{2}}$, y also

$\Rightarrow$ $[\pm\sqrt{\frac{23}{2}}, \pm\sqrt{\frac{23}{2}}, 0]$

– 4 points $\left[\pm\sqrt{\frac{23}{2}}, \pm\sqrt{\frac{23}{2}}, 0\right]$ .. all possibilities of signs

but $\sqrt{\frac{23}{2}} < 4$ (since $\frac{23}{2} < 16$), so they do not belong to M_1

Extrema on M_2 : Case 1 : ∇g_1 and ∇g_2 linearly dependent

2 pts

$\nabla g_1 = \alpha \nabla g_2 \Rightarrow \begin{matrix} 2x = 0 \\ 2y = \alpha \\ 2z = \alpha \end{matrix} \Rightarrow x = 0,\ y = z$

$y + z = 4 \Rightarrow y = z = 2$

$[0, 2, 2] \notin M_2$ as $2^2 + 2^2 = 8 \neq 23$

$\Rightarrow$ no points here

~~Case~~ $\nabla g_2 = \alpha \nabla g_1 \Rightarrow$ necessarily $\alpha \neq 0 \Rightarrow \nabla g_1 = \frac{1}{\alpha}\nabla g_2$ – impossible by the above

Case 2 : $\exists\, \lambda, \mu : \nabla f + \lambda \nabla g_1 + \mu \nabla g_2 = 0$

1 pt

$$\left\{\begin{aligned} y + \lambda\cdot 2x &= 0 \\ x + \lambda\cdot 2y + \mu &= 0 \\ \lambda\cdot 2z + \mu &= 0 \end{aligned}\right.$$

$x + 2\lambda(y - z) = 0 \Rightarrow y(y - z) - x^2 = 0$

$\Rightarrow x^2 = y^2 - yz$

plug into g_1, g_2 :

$y^2 - yz + y^2 + z^2 = 23$

$y + z = 4$

$z = 4 - y$

$y^2 - y(4 - y) + y^2 + (4 - y)^2 = 23$

$y^2 - 4y + y^2 + y^2 + 16 - 8y + y^2 = 23$

$4y^2 - 12y - 7 = 0$

$y = \frac{12 \pm \sqrt{144 + 7\cdot 16}}{8} =$

$7\cdot 16 = 112$, $144 + 112 = 256 = 16^2$

$= \frac{12 \pm 16}{8} = \begin{cases} -\frac{1}{2} \\ \frac{7}{2} \end{cases}$

---

3 pts

$y = -\frac{1}{2} \Rightarrow z = \frac{9}{2} \Rightarrow x^2 = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}$

$\left[\sqrt{\frac{5}{2}}, -\frac{1}{2}, \frac{9}{2}\right]$, $\left[-\sqrt{\frac{5}{2}}, -\frac{1}{2}, \frac{9}{2}\right]$

$\downarrow f$: $-\frac{\sqrt{5}}{2\sqrt{2}}$, $\downarrow f$: $\frac{\sqrt{5}}{2\sqrt{2}}$

$y = \frac{7}{2} \Rightarrow z = \frac{1}{2} \Rightarrow x^2 = \frac{49}{4} - \frac{7}{4} = \frac{42}{4} = \frac{21}{2}$

$\left[\sqrt{\frac{21}{2}}, \frac{7}{2}, \frac{1}{2}\right]$, $\left[-\sqrt{\frac{21}{2}}, \frac{7}{2}, \frac{1}{2}\right]$

$\downarrow f$: $\frac{7\sqrt{21}}{2\sqrt{2}}$, $\downarrow f$: $-\frac{7\sqrt{21}}{2\sqrt{2}}$

---

comparison : $-\frac{7\sqrt{21}}{2\sqrt{2}} < -\frac{\sqrt{5}}{2\sqrt{2}} < 0 < \frac{\sqrt{5}}{2\sqrt{2}} < \frac{7\sqrt{21}}{2\sqrt{2}}$

$\Rightarrow$ Max $\frac{7\sqrt{21}}{2\sqrt{2}}$ at $\left[\sqrt{\frac{21}{2}}, \frac{7}{2}, \frac{1}{2}\right]$, Min $-\frac{7\sqrt{21}}{2\sqrt{2}}$ at $\left[-\sqrt{\frac{21}{2}}, \frac{7}{2}, \frac{1}{2}\right]$

2 pts

PROBLEM B5 $\displaystyle\int \frac{2x^5+54}{x^5-2x^4+3x^3}\,dx$

① division: $(2x^5+54):(x^5-2x^4+3x^3)=2$ remainder $4x^4-6x^3+54$ $\Big\}$ $\displaystyle\frac{2x^5+54}{x^5-2x^4+3x^3}=2+\frac{4x^4-6x^3+54}{x^5-2x^4+3x^3}$

1 point

② partial fractions: $\displaystyle\frac{4x^4-6x^3+54}{x^5-2x^4+3x^3}=\frac{4x^4-6x^3+54}{x^3(x^2-2x+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2-2x+3}$

1 pt

$4x^4-6x^3+54=Ax^2(x^2-2x+3)+Bx(x^2-2x+3)+C(x^2-2x+3)+(Dx+E)x^3$

at x^4: $4=A+D$	
at x^3: $-6=-2A+B+E$	$\to A=\frac{1}{3}(2B-C)=\frac{1}{3}(24-18)=2$
at x^2: $0=3A-2B+C$	
at x^1: $0=3B-2C$	$\to B=\frac{2}{3}C=\frac{2}{3}\cdot 18=12$
at x^0: $54=3C$	$\Rightarrow C=18$

3 points

$D=4-A=2$, $E=2A-B-6=4-12-6=-14$

Wo ger: $\displaystyle\frac{2}{x}+\frac{12}{x^2}+\frac{18}{x^3}+\frac{2x-14}{x^2-2x+3}$

③ $\displaystyle\int\frac{2x-14}{x^2-2x+3}\,dx=\int\left(\frac{2x-2}{x^2-2x+3}+\frac{-12}{x^2-2x+3}\right)dx\overset{1\text{ pt}}{=}\lg(x^2-2x+3)-12\int\frac{1}{x^2-2x+3}\,dx$

1 point

1 pun

$\displaystyle-12\int\frac{1}{x^2-2x+3}\,dx=-12\int\frac{1}{(x-1)^2+2}\,dx=-\frac{12}{2}\int\frac{1}{\left(\frac{x-1}{\sqrt2}\right)^2+1}\,dx=$

2 points

$\displaystyle=-6\sqrt2\int\frac{\frac{1}{\sqrt2}}{\left(\frac{x-1}{\sqrt2}\right)^2+1}\,dx=-6\sqrt2\,\mathrm{arctg}\,\frac{x-1}{\sqrt2}$ (a IR)

The result:

$$\int \frac{2x^5+54}{x^5-2x^4+3x^3}\,dx \overset{c}{=} 2x \underbrace{+2\lg|x| - \frac{12}{x} - \frac{9}{x^2}}^{2\text{pts}} + \lg(x^2-2x+3) - 6\sqrt{2}\,\mathrm{arctg}\,\frac{x-1}{\sqrt{2}}$$

on $(-\infty,0)$ and on $(0,+\infty)$

2 points

1 point