PROBLEM A1

Gauss elimination: 1 point

$$\left(\begin{array}{cccc|ccc} 2 & 1 & 2 & 5 & -3 & 1 & 1 \\ 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 5 & 1 & 2 & 2 & 3 & 1 & 1 \\ 5 & 2 & 2 & 1 & 4 & 1 & 3 \\ \hline \multicolumn{4}{c|}{A} & b_1 & b_2 & b_3 \end{array}\right) \sim \begin{array}{c} \\ -2\cdot \\ -5\cdot \\ -5\cdot \end{array} \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 2 & 1 & 2 & 5 & -3 & 1 & 1 \\ 5 & 1 & 2 & 2 & 3 & 1 & 1 \\ 5 & 2 & 2 & 1 & 4 & 1 & 3 \end{array}\right) \sim$$

4 pts

$$\sim \begin{array}{c} \\ \\ -3\cdot \\ -3 \end{array} \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 0 & -3 & -2 & -5 & 5 & -1 & -3 \\ 0 & -9 & -8 & -23 & 23 & -4 & -9 \\ 0 & -8 & -8 & -24 & 24 & -4 & -7 \end{array}\right) \sim \begin{array}{c} \\ \\ \\ +3\cdot \end{array} \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 0 & -3 & -2 & -5 & 5 & -1 & -3 \\ 0 & 0 & -2 & -8 & +8 & -1 & 0 \\ 0 & 1 & -2 & -9 & 9 & -1 & 2 \end{array}\right)$$

$$\sim \begin{array}{c} \\ -4\cdot \\ \\ \\ \end{array} \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 0 & 0 & -8 & -32 & 32 & -4 & 3 \\ 0 & 0 & -2 & -8 & +8 & -1 & 0 \\ 0 & 1 & -2 & -9 & 9 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & -8 & -8 & -1 & 0 \\ 0 & 1 & 2 & -9 & 9 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 5 & -4 & 1 & 2 \\ 0 & 1 & -2 & -9 & 9 & -1 & 2 \\ 0 & 0 & -2 & -8 & +8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

Solution for b_1:

$-2x_3 - 8x_4 = +8$

$x_3 = -4 - 4x_4$

$x_4 = t, \quad x_3 = -4 - 4t$

$x_2 - 2x_3 - 9x_4 = 9$ 2 points

$x_2 = 9 + 2x_3 + 9x_4 =$

$= 9 - 8 - 8t + 9t =$

$1 + t$

$x_1 + 2x_2 + 2x_3 + 5x_4 = -4$

$x_1 = -4 - 2x_2 - 2x_3 - 5x_4$

$= -4 - 2 - 2t + 8 + 8t - 5t$

$= 2 + t$

$\Rightarrow [2+t, 1+t, -4-4t, t]$

$t \in \mathbb{R}$

Solution for b_2:

$-2x_3 - 8x_4 = -1$

$x_4 = t, \quad x_3 = \frac{1}{2} - 4t$

$x_2 - 2x_3 - 9x_4 = -1$

$x_2 = -1 + 2x_3 + 9x_4 =$

$= -1 + 1 - 8t + 9t = t$

$x_1 + 2x_2 + 2x_3 + 5x_4 = 1$ 2 points

$x_1 = 1 - 2x_2 - 2x_3 - 5x_4$

$= 1 - 2t - 1 + 8t - 5t$

$= t$

$\Downarrow$

$[t, t, \frac{1}{2} - 4t, t]$

$t \in \mathbb{R}$

Solution for b_3

$0 = 3$

no solution

1 point

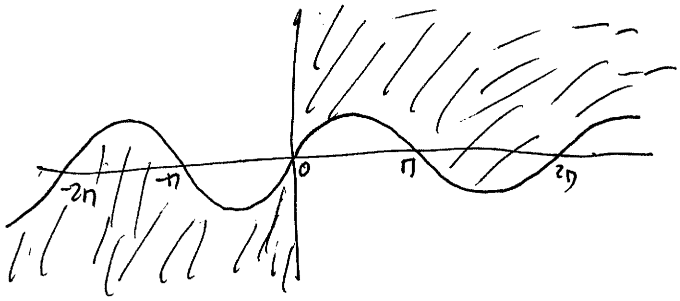
PROBLEM A2

$f(x,y) = \sqrt{x(y-\sin x)}$

D_f: $x(y-\sin x) \geq 0$

2 points $\Rightarrow D_f = \{[x,y] \in \mathbb{R}^2 : \ x \geq 0 \ \& \ y \geq \sin x \ \text{ or } \ x \leq 0 \ \& \ y \leq \sin x\}$

Pictures



including the boundary

1 point

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{2\sqrt{x(y-\sin x)}} \cdot (y - \sin x - x \cdot \cos x)\\ \frac{\partial f}{\partial y} &= \frac{1}{2\sqrt{x(y-\sin x)}} \cdot x\end{aligned}\right\} \text{ if } \begin{aligned}& x>0 \ \& \ y > \sin x\\ \text{or } & x<0 \ \& \ y < \sin x\end{aligned}$$

2 points

1 point

In the remaining points: $x = 0$, i.e. $[0,y], y \in \mathbb{R}$

- $\frac{\partial f}{\partial x}(0,y)$ has no sense (no horizontal segment around $[0,y]$ in D_f)
- $\frac{\partial f}{\partial y}(0,y) = 0$ since $f(0,y) = 0$ for $y \in \mathbb{R}$

1 point

$x \neq 0, y = \sin x$: $\frac{\partial f}{\partial y}(x, \sin x)$ has no sense (no vertical segment around $[x, \sin x]$ in D_f)

$\frac{\partial f}{\partial x}(x, \sin x)$ should be computed for $x = \frac{\pi}{2} + 2k\pi, k \geq 0$ and $x = -\frac{\pi}{2} + 2k\pi, k \leq 0$

(otherwise no horizontal segment around the point in D_f)

1pt

$k \geq 0$, $k \in \mathbb{Z}$: $\frac{\partial f}{\partial x}\left(\frac{\pi}{2} + 2k\pi, 1\right) = \lim\limits_{h\to 0} \frac{\sqrt{\left(\frac{\pi}{2}+2k\pi+h\right)\left(1-\sin\left(\frac{\pi}{2}+2k\pi+h\right)\right)} - 0}{h} =$

$= \sqrt{\frac{\pi}{2}+2k\pi} \cdot \lim\limits_{h\to 0} \frac{\sqrt{1-\sin\left(\frac{\pi}{2}+h\right)}}{h} = \sqrt{\frac{\pi}{2}+2k\pi} \lim\limits_{h \to 0} \operatorname{sgn} h \cdot \underbrace{\sqrt{\frac{1-\cos h}{h^2}}}_{\to \frac{1}{\sqrt{2}}} = \begin{cases} \xrightarrow{h\to 0+} \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}+2k\pi} \\ \xrightarrow{h \to 0-} -\frac{1}{2}\sqrt{\frac{\pi}{2}+2k} \end{cases}$

doesn't exist

1pt

$k \leq 0$, $k \in \mathbb{Z}$ $\frac{\partial f}{\partial x}\left(-\frac{\pi}{2}+2k\pi, -1\right) = \lim\limits_{h\to 0} \frac{\sqrt{\left(-\frac{\pi}{2}+2k\pi+h\right)\left(-1-\sin\left(-\frac{\pi}{2}+2k\pi+h\right)\right)}}{h} =$

$= \sqrt{\frac{\pi}{2}-2k\pi} \cdot \lim\limits_{h\to 0} \frac{\sqrt{1+\sin\left(-\frac{\pi}{2}+h\right)}}{h} = \sqrt{\frac{\pi}{2}-2k\pi} \cdot \lim\limits_{h\to 0} \frac{\sqrt{1-\cos h}}{h}$ – doesn't exist as above

1pt

PROBLEM A3 $\underbrace{\operatorname{arctg}(2x+y)+\sin(y^2-x^2)+\frac{\pi}{4}}_{F(x,y)}=0$, $[-1,1]$

3pt
- $F \in C^\infty(\mathbb{R}^2)$ ... composed of C^∞ functions
- $F(-1,1)=\operatorname{arctg}(-2+1)+\sin(1-1)+\frac{\pi}{4}=\operatorname{arctg}(-1)+0+\frac{\pi}{4}=-\frac{\pi}{4}+\frac{\pi}{4}=0$
- $\frac{\partial F}{\partial y}(-1,1)=\left(\frac{1}{1+(2x+y)^2}\cdot 1+\cos(y^2-x^2)\cdot 2y\right)_{\substack{x=-1\\y=1}}=\frac{1}{1+(-2+1)^2}\cdot 1+\cos 0\cdot 2=$
  $=\frac{1}{2}+2=\frac{5}{2}\neq 0$

Hence, by IFT there exists a C^∞ function f with the required properties
In part.: $\operatorname{arctg}(2x+f(x))+\sin(f(x)^2-x^2)+\frac{\pi}{4}=0$ on a nbhd of -1 1pt

~~plug in $x=-1$, use $f(-1)=1$:~~ ~~arctg~~

Differentiate: $\frac{1}{1+(2x+f(x))^2}\cdot(2+f'(x))+\cos(f(x)^2-x^2)\cdot(2f(x)f'(x)-2x)=0$ 1pt

plug in $x=-1$, use $f(-1)=1$: $\frac{1}{1+(-2+1)^2}(2+f'(-1))+\cos(1-1)(2\cdot 1\cdot f'(-1)+2)=0$
1pt
$1+\frac{1}{2}f'(-1)+2f'(-1)+2=0 \Rightarrow \frac{5}{2}f'(-1)=-3 \Rightarrow f'(-1)=-\frac{6}{5}$

Tangent line: $y=1-\frac{5}{6}(x+1)$ 1pt

Second derivative: $-\frac{1}{(1+(2x+f(x))^2)^2}\cdot 2(2x+f(x))\cdot(2+f'(x))^2+\frac{1}{1+(2x+f(x))^2}\cdot f''(x)$
2pt
$-\sin(f(x)^2-x^2)(2f(x)f'(x)-2x)^2+\cos(f(x)^2-x^2)\cdot(2f'(x)\cdot f'(x)+2f(x)\cdot f''(x)-2)=0$

Plug in $x=-1$, use $f(-1)=1$, $f'(-1)=-6/5$: $-\frac{1}{(1+1)^2}\cdot 2\cdot(-2+1)\cdot\left(2-\frac{6}{5}\right)^2+\frac{1}{1+1}\cdot f''(-1)-\sin 0\cdot(\dots)$
$+\cos 0\cdot\left(2\cdot\left(-\frac{6}{5}\right)^2+2\cdot 1\cdot f''(-1)-2\right)=0$

$-$ [illegible] $+\frac{1}{2}\cdot\frac{16}{25}+\frac{1}{2}f''(1)+2\cdot\frac{36}{25}+2f''(-1)-2=0$

$\frac{5}{2}f''(-1)+\frac{8+72-50}{25}=0$ 1pt

$\frac{5}{2}f''(-1)+\frac{6}{5}=0 \Rightarrow f''(-1)=-\frac{12}{25}$

PROBLEM A4 $f(x,y,z) = xz$

$$M = \{ [x,y,z] \in \mathbb{R}^3 : x^2+2y^2 \leq 3 , z^2-y^2 = 1 \}$$

① Existence of extrema: f is cts on $\mathbb{R}^3$

M is clearly closed ("≤" and "=")

M is bounded: By $x^2+2y^2 \leq 3$ we get $x^2 \leq 3 \Rightarrow |x| \leq \sqrt{3}$

$2y^2 \leq 3 \Rightarrow |y| \leq \sqrt{\frac{3}{2}}$

further, by $z^2-y^2=1$ we have

$z^2 = 1+y^2 \leq 1+\frac{3}{2} \Rightarrow |z| \leq \sqrt{\frac{5}{2}}$

2pts

$\Rightarrow$ M compact

So, extrema exist provided $M \neq \emptyset$.

② $M = M_1 \cup M_2$ $M_1 = \{ [x,y,z] : x^2+2y^2 < 3 , z^2-y^2=1 \}$

$M_2 = \{ [x,y,z] : x^2+2y^2=3 , z^2-y^2=1 \}$

1 point

Denote $g_1(x,y,z) = x^2+2y^2-3$, $g_2(x,y,z) = z^2-y^2-1$

For future use: $Df = [z,0,x]$, $Dg_1 = [2x,4y,0]$, $Dg_2 = [0,-2y,2z]$

③ **ON M_1:** Either $Dg_2 = 0$ ($\Rightarrow y = z = 0$, not possible since $z^2-y^2=1$ on M_1) 1pt

or $\exists \lambda : Df + \lambda \cdot Dg_2 = 0$

$$z + \lambda \cdot 0 = 0$$
$$0 \mp \lambda \cdot 2y = 0$$
$$x + \lambda \cdot 2z = 0$$

$\Rightarrow z = 0$

$\Rightarrow x = 0$, hence $x = z = 0$

$z^2-y^2=1 \Rightarrow -y^2 = 1$ no solution

1pt

$\Rightarrow$ no extrema on M_1

2pts

④ **ON M_2:** **Case 1:** Dg_1 and Dg_2 linearly dependent

2pts

$Dg_1 = \alpha \cdot Dg_2 \Rightarrow 2x = \alpha \cdot 0,\ 4y = \alpha \cdot (-2y)\ \ 0 = \alpha \cdot 2z$

if $\alpha = 0 \Rightarrow x = y = 0$ not in the set M_2, since $x^2+2y^2=3$

if $\alpha \neq 0 \Rightarrow x = 0 = z$: contradicts $z^2 - y^2 = 1$
$(\Rightarrow -y^2 = 1)$

$Dg_2 = \alpha \cdot Dg_1$ ... $\alpha \neq 0 \Rightarrow$ impossible by the above (then $Dg_1 = \frac{1}{\alpha} Dg_2$)

- $\alpha = 0 \Rightarrow Dg_2 = 0 \Rightarrow y = z = 0$ impossible in M_2 as $y^2 - z^2 = -1$

So, no points from Case 1

Case 2: $Df + \lambda Dg_1 + \mu Dg_2 = 0$ for suitable $\lambda, \mu \in \mathbb{R}$

1pt

$$\begin{cases} z + \lambda \cdot 2x + \mu \cdot 0 = 0 \\ 0 + \lambda \cdot 4y + \mu \cdot (-2y) = 0 \\ x + \lambda \cdot 0 + \mu \cdot 2z = 0 \end{cases}$$

$\rightarrow 2y(2\lambda - \mu) = 0 \Rightarrow y = 0$ or $\mu = 2\lambda$

- $y = 0$: $x^2 = 3,\ z^2 = 1$

$[\sqrt{3}, 0, 1] \xmapsto{f} \sqrt{3}$

$[-\sqrt{3}, 0, 1] \xmapsto{f} -\sqrt{3}$

$[\sqrt{3}, 0, -1] \xmapsto{f} -\sqrt{3}$

$[-\sqrt{3}, 0, -1] \xmapsto{f} \sqrt{3}$

$\mu = 2\lambda$: $z + \lambda \cdot 2x = 0 \quad (1)$
$x + 4\lambda \cdot z = 0 \quad (2)$

$2z \cdot (1) - x \cdot (2)$:
$2z^2 - x^2 = 0 \Rightarrow x^2 = 2z^2$
$\Rightarrow x = \sqrt{2} \cdot z$ or $x = -\sqrt{2} z$

$2z^2 + 2y^2 = 3 \Rightarrow z^2 + y^2 = \frac{3}{2}$
$z^2 - y^2 = 1$

$\rightarrow z^2 = \frac{5}{4},\ y^2 = \frac{1}{4}$

$\Downarrow$
$x^2 = \frac{5}{2}$

3 points

⑤ Comparing the points:

$\sqrt{3} \overset{?}{<} \frac{5}{2\sqrt{2}}$

$3 < \frac{25}{8}$

$24 < 25$

YES

$\Rightarrow$ max $\frac{5}{2\sqrt{2}}$

min $-\frac{5}{2\sqrt{2}}$

at the points $\rightarrow$

2 points

all possible signs $\Rightarrow$ 8 points

$\left[\sqrt{\tfrac{5}{2}}, \tfrac{1}{2}, \tfrac{\sqrt{5}}{2}\right]$, $\left[\sqrt{\tfrac{5}{2}}, \tfrac{1}{2}, -\tfrac{\sqrt{5}}{2}\right]$

$\left[\sqrt{\tfrac{5}{2}}, -\tfrac{1}{2}, \tfrac{\sqrt{5}}{2}\right]$, $\left[\sqrt{\tfrac{5}{2}}, -\tfrac{1}{2}, -\tfrac{\sqrt{5}}{2}\right]$

$\left[-\sqrt{\tfrac{5}{2}}, \tfrac{1}{2}, \tfrac{-\sqrt{5}}{2}\right]$, $\left[-\sqrt{\tfrac{5}{2}}, \tfrac{1}{2}, \tfrac{\sqrt{5}}{2}\right]$

$\left[-\sqrt{\tfrac{5}{2}}, -\tfrac{1}{2}, \tfrac{-\sqrt{5}}{2}\right]$, $\left[-\sqrt{\tfrac{5}{2}}, (-\tfrac{1}{2}), \tfrac{\sqrt{5}}{2}\right]$

$\downarrow f$ $\frac{5}{2\sqrt{2}}$

$\downarrow f$ $-\frac{5}{2\sqrt{2}}$

PROBLEM A5 $\int \frac{7x^4}{(x^2-1)(x^2+2x+4)} dx$

① It is a rational fraction. We should divide the polynomials

$$(x^2-1)(x^2+2x+4) = x^4+2x^3+4x^2-x^2-2x-4 = x^4+2x^3+3x^2-2x-4$$

$$7x^4 : (x^4+2x^3+3x^2-2x-4) = 7$$

1pt

remainder: $-14x^3-21x^2+14x+28$

$$\Rightarrow \frac{7x^4}{(x^2-1)(x^2+2x+4)} = 7 + \frac{-14x^3-21x^2+14x+28}{(x^2-1)(x^2+2x+4)}$$

② Partial fractions

1pt

$$\frac{-14x^3-21x^2+14x+28}{(x-1)(x+1)(x^2+2x+4)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+2x+4} = (*)$$

$$-14x^3-21x^2+14x+28 = A(x+1)(x^2+2x+4) + B(x-1)(x^2+2x+4) + (Cx+D)(x^2-1)$$

$x=1$: $-14-21+14+28 = A\cdot 2\cdot 7 \Rightarrow 7 = 14A \Rightarrow A = \frac{1}{2}$

$x=-1$: $14-21-14+28 = B\cdot(-2)\cdot(1-2+4) \Rightarrow 7 = -6B \Rightarrow B = -\frac{7}{6}$

coef. at x^3: $-14 = A+B+C = \frac{1}{2} - \frac{7}{6} + C = C - \frac{2}{3}$

$$\Rightarrow C = -14 + \frac{2}{3} = -\frac{40}{3}$$

coef. at x^0: $28 = 4A - 4B - D = 2 + \frac{14}{3} - D = \frac{20}{3} - D$

$$D = -28 + \frac{20}{3} = -\frac{64}{3}$$

3 pts

$\Rightarrow$

$$(*) = \frac{\frac{1}{2}}{x-1} - \frac{\frac{7}{6}}{x+1} + \frac{-\frac{40}{3}x - \frac{64}{3}}{x^2+2x+4}$$

③ Integration of the third fraction

1pt

$$\int \frac{-\frac{40}{3}x-\frac{64}{3}}{t^2+2t+4}\,dt = -\frac{8}{3}\int\frac{5t+8}{t^2+2t+4} = -\frac{8}{3}\int\left(\frac{5}{2}\frac{2t+2}{t^2+2t+4}+\frac{8-5}{t^2+2t+4}\right)dt$$

1pt

$$= -\frac{20}{3}\lg(t^2+2t+4) \underbrace{-8\int\frac{1}{t^2+2t+4}dt}$$

3pts

$$= -8\int\frac{1}{(t+1)^2+3}dt = -\frac{8}{3}\int\frac{1}{\left(\frac{t+1}{\sqrt{3}}\right)^2+1} = -\frac{8}{3}\cdot\sqrt{3}\int\frac{\frac{1}{\sqrt{3}}}{\left(\frac{t+1}{\sqrt{3}}\right)^2+1}dt$$

$$\overset{c}{=} -\frac{8}{\sqrt{3}}\operatorname{arctg}\left(\frac{t+1}{\sqrt{3}}\right) \text{ on } \mathbb{R}$$

④ So, the result is :

1pt 1pt

$$\int\frac{7t^4}{(t^2-1)(t^2+2t+4)}dt \overset{c}{=} 7t+\frac{1}{2}\lg|t-1|-\frac{7}{6}\lg|t+1|-\frac{20}{3}\lg(t^2+2t+4)$$
$$-\frac{8}{\sqrt{3}}\operatorname{arctg}\frac{t+1}{\sqrt{3}}$$

1pt

on each of the intervals:

$(-\infty,-1)$

$(-1,1)$

$(1,+\infty)$

2pts