FEW REMARKS ON STRUCTURE OF CERTAIN SPACES OF MEASURES

In this note we study the structure of spaces of measures on spaces K_B introduced in [Ka3], which forms the previous chapter of this thesis. In [Ka3] it is proved that under certain assumptions on B (some additional axioms of the set theory are needed) the space K_B belongs to the Stegall class $\mathcal{S}$ but is not fragmentable. This is related to a well known open problem, whether there is a Stegall Banach space (i.e., a space X such that $(X^\star, w^\star)$ belongs to $\mathcal{S}$) whose dual is not fragmentable (in the $w^\star$-topology). In this context a natural question is whether the dual unit ball $\left(B_{\mathcal{C}(K_B)^\star}, w^\star\right)$ belongs to the class $\mathcal{S}$, too. We present here some partial positive results in this direction. This is inspired by the following characterization due to C.Stegall.

Theorem A([S]). *Let K be a compact Hausdorff space. Then the following conditions are equivalent.*

(i) K is Stegall with respect to completely regular spaces (i.e. every minimal usco from a completely regular Baire space into K is singlevalued at points of a residual set).

(ii) For every complete metric space M and every closed subset $F \subset K \times M$ the set F contains a dense completely metrizable subspace.

We show that, under certain additional set-theoretical assumptions, there is a set B such that K_B is Stegall and non-fragmentable and, moreover, each ccc closed subset of $\mathcal{M}_+(K_B)$ contains a dense completely metrizable subset (Example). However, this does not cover all closed subsets of $\mathcal{M}_+(K_B)$ since there are closed subsets of $\mathcal{M}_+(K_B)$ which are even "locally non-ccc" (by Proposition 4).

In proving our Example we need two results which may be of an independent interest – Theorem 1 on "Baire-property-additive" systems in compact spaces and Theorem 2 on descriptive properties of sets of measures having an atom in a given set. Theorem 1 is proved in Section 1. Section 2 is devoted to some general facts on spaces of measures. In Section 3 we prove Theorem 2 and in the last section we prove Example using both theorems and some auxiliary results on structure of the space $\mathcal{M}_+(K_B)$.

1. A theorem on Baire-property-additive systems.

In this section we prove a result on "Baire-property-additive" families in certain compact spaces. We will use this result in Section 4 to prove Proposition 8.

Before stating the result let us recall that by $c(X)$, where X is a topological space, we denote the *Suslin number* of X, i.e.

$$c(X) = \sup \{\operatorname{card} \mathcal{E} \mid \mathcal{E} \text{ is a disjoint family of nonempty open sets in } X\}.$$

An uncountable cardinal is called *weakly inaccessible* if it is regular and limit. Also let us recall that a subset A of a topological space X is said to have *restricted Baire property* if, for every $H \subset X$, the set $A \cap H$ has the Baire property in H.

Theorem 1. *(i) Let K be a compact Hausdorff space such that $c(K) < 2^{\aleph_0}$ and $\operatorname{card} K$ is less than the least weakly inaccessible cardinal. Then, whenever $\mathcal{E}$ is a point-finite family of meager sets in K such that the union of every subfamily has the Baire property in K, the union $\bigcup \mathcal{E}$ is meager in K, too.*

(ii) Let K be a compact Hausdorff space of cardinality $\leq 2^{\aleph_0}$ and $\mathcal{E}$ a point-finite family of meager subsets of K such that $\bigcup \mathcal{E}'$ has the restricted Baire property for every $\mathcal{E}' \subset \mathcal{E}$. Then $\bigcup \mathcal{E}$ is meager.

Proof. (i) By [Fr,Corollary 7D] it is enough to consider disjoint families. Let $\mathcal{T} = \{G \setminus N \mid G \text{ open}, N \text{ meager in } K\}$. By [Ka2, Lemma 2] $\mathcal{T}$ is a topology. Put $H = (K, \mathcal{T})$. By [Fr, Lemma 2O] H is weakly α-favorable, it is easy to see (and follows from [Ka1, Lemma 6.1(h)]) that $c(H) = c(K) < 2^{\aleph_0}$ and by [Ka2, Lemma 2(c)] every set with the Baire property in H is of the form $F \cap G$ with F closed and G open in H. So by [Ka1, Theorem 5.5(i)] the conclusion holds if $\operatorname{card} \mathcal{E}$ is less than the least weakly inaccessible cardinal, but $\operatorname{card} \mathcal{E}$ cannot have greater cardinality than $\operatorname{card} K$ for $\mathcal{E}$ is disjoint. This finishes the proof.

(ii) By [Fr,Theorem 4D(b)] it is enough to consider disjoint families. Now, let $\mathcal{E}$ be a disjoint family of meager sets in K, such that the union of every subfamily has the restricted Baire property. Since $\mathcal{E}$ has clearly cardinality at most $2^{\aleph_0}$, in particular, less than the least measurable cardinal, by [P] we get that $\bigcup \mathcal{E}$ is meager. $\square$

In fact, we will use only the part (ii) of this theorem. But we give here part (i), too, for two reasons. Firstly, the part (i) deals with the Baire property, which is a more general notion than that of restricted Baire property, so we find interesting to compare it with part (ii). And secondly, using part (i) we can get the same example with only a bit stronger set-theoretical assumptions than using part (ii).

2. Some general facts on spaces of measures on compact spaces.

In this section we collect several general facts which we will use in Section 4.

If K is a compact Hausdorff space, by $\mathcal{C}(K)$ we denote the Banach space of all continuous functions on K, by $\mathcal{M}(K)$ the space of all finite signed Radon measures on K which is canonically identified with the dual of $\mathcal{C}(K)$. By $\mathcal{M}_+(K)$ we denote the cone of positive measures from $\mathcal{M}(K)$, by $P(K)$ the probability Radon measures on K. The spaces $\mathcal{M}_+(K)$ and $P(K)$ are considered (unless specified otherwise) with the $w^\star$-topology inherited from $\mathcal{C}(K)^\star$. We will often use the fact (see [Ko]) that the sets $\{\mu \in \mathcal{M}_+(K) \mid \mu(K) < c\}$ for $c > 0$ and $\{\mu \in \mathcal{M}_+(K) \mid \mu(G) > c\}$ for $G \subset K$ open and $c > 0$ form a subbase for the $w^\star$ topology on $\mathcal{M}_+(K)$.

Proposition 1. *Let K, L be compact Hausdorff spaces, $F : K \to L$ a continuous surjection. Then the mapping $\tilde{F} : \mathcal{C}(L) \to \mathcal{C}(K)$ defined by $\tilde{F}(f) = f \circ F$ is a linear isometry of $\mathcal{C}(L)$ into $\mathcal{C}(K)$. Consider the adjoint operator $\tilde{F}^\star : \mathcal{M}(K) \to \mathcal{M}(L)$. Then $\tilde{F}^\star$ is continuous when both spaces are equipped with the $w^\star$ topology, maps positive measures onto positive measures, is norm-preserving on positive measures and for any $\mu \in \mathcal{M}(K)$ its image $\tilde{F}^\star(\mu)$ is the image of μ by F (denoted $F(\mu)$), that means $\tilde{F}^*(\mu)(B) = \mu(F^{-1}(B))$ for every $B \subset L$ Borel.*

Proof. It is obvious that $\tilde{F}$ is a linear isometry. Recall that $\tilde{F}^\star$ is defined by $\tilde{F}^\star(\mu)(f) = \mu(f \circ F)$ for $\mu \in \mathcal{M}(K)$ and $f \in \mathcal{C}(L)$. So clearly $\tilde{F}^\star(\mu)$ is positive whenever μ is positive and $\tilde{F}^\star$ is $w^\star \to w^\star$ continuous, norm-preserving on positive measures (since if $\mu \geq 0$ then $\|\mu\| = \mu(1)$). Moreover, $\tilde{F}^\star\left(\mathcal{M}_+(K)\right)$ contains all positive measures on L supported by a finite set (by the surjectivity of F). So it follows, by $w^\star$-density of finitely supported measures and $w^\star$ compactness of the unit ball, that $\tilde{F}^\star$ maps $\mathcal{M}_+(K)$ onto $\mathcal{M}_+(L)$. The identification of $\tilde{F}^\star(\mu)$ with $F(\mu)$ easily follows from the substitution theorem. $\square$

Proposition 2. *Let K be a compact Hausdorff space. Then $P(K)$ contains a homeomorphic copy of $K^{\mathbb{N}}$.*

Proof. For a sequence $(k_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ let us define $g((k_n)) = \sum\limits_{n\in\mathbb{N}} \frac{2}{3^n}\delta_{k_n}$. It is standard to check that $g : K^{\mathbb{N}} \to P(K)$ is continuous since the topology on $K^{\mathbb{N}}$ is that of pointwise convergence. Moreover, g is one-to-one. To see it take $(k_n) \neq (l_n)$ two different elements of $K^{\mathbb{N}}$. There is a minimal m such that $k_m \neq l_m$. Find $f : K \to [0,1]$ continuous with $f(k_m) = 0$ and $f(l_m) = 1$. Then $g((l_n))(f) - g((k_n))(f) \geq \frac{1}{3^m}$. Hence g is one-to-one and so it is a homeomorphism onto its image. $\square$

Proposition 3. *Let K be a compact Hausdorff space.*

(1) If $\mathcal{B}$ is a basis for the topology of K then the family

$$\begin{aligned}\mathcal{B}_M = \{\{\mu \in \mathcal{M}_+(K) \mid \mu(K) < p, \mu\left(B_1^i \cup \cdots \cup B_{k_i}^i\right) > q_i,\ i = 1, \ldots, n\} \mid \\ B_j^i \in \mathcal{B},\ j = 1, \ldots, k_i, k_i \in \mathbb{N}, q_i \in \mathbb{Q}, i = 1, \ldots, n, n \in \mathbb{N}, p \in \mathbb{Q}\}\end{aligned}$$

forms a basis of $\mathcal{M}_+(K)$. In particular $w(K) + \aleph_0 = w\left(\mathcal{M}_+(K)\right) + \aleph_0$.

(2) If $\mu \in \mathcal{M}_+(K)$ and $\mathcal{P}$ is a family of open subsets of K with the property that for any $U \subset K$ open there is $\mathcal{C} \subset \mathcal{P}$ such that $\bigcup\mathcal{C} \subset U$ and $\mu\left(U \setminus \bigcup\mathcal{C}\right) = 0$ then the collection

$$\begin{aligned}&\mathcal{P}_M = \{\{\nu \in \mathcal{M}_+(K) \mid \\ &\nu(K) < \mu(K) + \frac{1}{m}, \nu\left(P_1^i \cup \cdots \cup P_{k_i}^i\right) > \mu\left(P_1^i \cup \cdots \cup P_{k_i}^i\right) - \frac{1}{q_i}, i = 1, \ldots, n\} \mid \\ &\qquad P_j^i \in \mathcal{P},\ j = 1, \ldots, k_i, k_i, q_i \in \mathbb{N}, i = 1, \ldots, n, n, m \in \mathbb{N}\}\end{aligned}$$

forms a neighborhood basis of μ. In particular the character of $\mathcal{M}_+(K)$ at μ is at most $\operatorname{card}\mathcal{P} + \aleph_0$.

Proof. (1) Let $\mu \in \mathcal{M}_+(K)$ and U be an open set containing μ. There are $G_1, \ldots, G_n$ open subsets of K and $c, \varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$\mu \in \{\nu \in \mathcal{M}_+(K) \mid \nu(K) < c, \nu(G_i) > \varepsilon_i, i = 1, \ldots, n\} \subset U.$$

Choose a rational number p such that $\mu(K) < p < c$. For $i = 1, \ldots, n$ choose q_i rational such that $\mu(G_i) > q_i > \varepsilon_i$. By the regularity of μ there is $H_i \subset G_i$ compact with $\mu(H_i) > q_i$. Since $\mathcal{B}$ is a basis and H_i is compact, there are $B_1^i, \ldots, B_{k_i}^i \in \mathcal{B}$ satisfying $H_i \subset B_1^i \cup \cdots \cup B_{k_i}^i \subset G_i$. Then clearly

$$\mu \in \{\nu \in \mathcal{M}_+(K) \mid \nu(K) < p, \nu\left(B_1^i \cup \cdots \cup B_{k_i}^i\right) > q_i,\ i = 1, \ldots, n\} \subset U,$$

which shows that $\mathcal{B}_M$ is a basis. Now, it is clear that $\operatorname{card}\mathcal{B}_M \leq \operatorname{card}\mathcal{B} + \aleph_0$, so $w\left(\mathcal{M}_+\left(K\right)\right) + \aleph_0 \leq w(K) + \aleph_0$. The inverse inequality follows from the fact that $\mathcal{M}_+\left(K\right)$ contains a copy of K.

(2) Let U be an open set containing μ. Then there are $G_1, \ldots, G_n$ open subsets of K and $c, \varepsilon_1, \ldots, \varepsilon_n > 0$ such that

$$\mu \in \{\nu \in \mathcal{M}_+\left(K\right) \mid \nu(K) < c, \nu(G_i) > \varepsilon_i, i = 1, \ldots, n\} \subset U.$$

Choose a natural number m such that $\mu(K) + \frac{1}{m} < c$. Now let $i \in \{1, \ldots, n\}$. There is $\mathcal{C} \subset \mathcal{P}$ with $\bigcup \mathcal{C} \subset G_i$ and $\mu\left(G_i \setminus \bigcup \mathcal{C}\right) = 0$. Find $q_i \in \mathbb{N}$ such that $\mu\left(G_i\right) - \frac{2}{q_i} > \varepsilon_i$. By the regularity of μ (similarly as in (1)) we find $P_1^i, \ldots, P_{k_i}^i \in \mathcal{C}$ such that $\mu\left(P_1^i \cup \cdots \cup P_{k_i}^i\right) > \mu\left(G_i\right) - \frac{1}{q_i}$. Then clearly

$$\begin{aligned}\mu \in \Big\{\nu \in \mathcal{M}_+\left(K\right) \mid \nu(K) < \mu(K) + \frac{1}{m}, \\ \nu\left(P_1^i \cup \cdots \cup P_{k_i}^i\right) > \mu\left(P_1^i \cup \cdots \cup P_{k_i}^i\right) - \frac{1}{q_i}, i = 1, \ldots, n\Big\} \subset U,\end{aligned}$$

which completes the proof. $\square$

3. Descriptive properties of certain sets of measures.

The main goal of this section is to prove the following theorem which we will need in Section 4 in the proof of Proposition 8.

Theorem 2. *If K is a compact Hausdorff space and $A \subset K$ is Borel (Suslin-$\mathcal{F}$, Suslin-$\mathcal{B}$, co-Suslin-$\mathcal{F}$, co-Suslin-$\mathcal{B}$) then so are the sets $\{\mu \in \mathcal{M}_+(K) \mid (\exists a \in A)(\mu(\{a\}) > c)\}$ and $\{\mu \in \mathcal{M}_+(K) \mid (\exists a \in A)(\mu(\{a\}) \geq c)\}$ for every $c \geq 0$.*

We prove this theorem in two lemmas. To this end we fix the following notation.

Let K be a compact Hausdorff space, and $\mathcal{M}_+(K)$ be the space of all positive finite Radon measures on K endowed with the $w^\star$ topology. For $A \subset K$ arbitrary, $p, s > 0$ we put

$$\begin{aligned} F_{A,p}^s &= \{\mu \in \mathcal{M}_+(K) \mid (\exists B \subset A \text{ finite})\,(\mu(B) \geq s \;\&\; (\forall b \in B)\,(\mu(\{b\}) \geq p))\} \\ G_{A,p}^s &= \{\mu \in \mathcal{M}_+(K) \mid (\exists B \subset A \text{ finite})\,(\mu(B) > s \;\&\; (\forall b \in B)\,(\mu(\{b\}) \geq p))\} \end{aligned}$$

Lemma 1. *Let $A \subset K$ be arbitrary, $p, s > 0$. Then the following hold.*

(1) $G_{A,p}^s = \bigcup_{n\in\mathbb{N}} F_{A,p}^{s+\frac{1}{n}}, \quad F_{A,p}^s = \bigcap_{n\in\mathbb{N}} G_{A,p}^{s-\frac{1}{n}}$.

(2) If A is closed then $F_{A,p}^s$ is closed (and $G_{A,p}^s$ is $\mathcal{F}_\sigma$).

(3) If $B \subset A$ then $G_{A\setminus B,p}^s = \bigcup_{t>s, t\in\mathbb{Q}} \left(G_{A,p}^t \setminus G_{B,p}^{t-s}\right)$.

(4) If $\mathcal{A}$ is a collection of subsets of K which is closed to finite unions then $G_{\bigcup \mathcal{A},p}^s = \bigcup\left\{G_{A,p}^s \mid A \in \mathcal{A}\right\}$. The analogue holds for $F_{\bigcup \mathcal{A},p}^s$.

(5) If $\mathcal{A}$ is a collection of subsets of K which is closed to finite intersections then $G_{\bigcap \mathcal{A},p}^s = \bigcap\left\{G_{A,p}^s \mid A \in \mathcal{A}\right\}$. The analogue holds for $F_{\bigcap \mathcal{A},p}^s$.

Proof. The assertion (1) is obvious. To see (2) let A be closed and μ_α be a net in $F_{A,p}^s$ converging to $\mu \in \mathcal{M}_+(K)$. By the definition of the $w^\star$ topology we have

$\mu_\alpha(K) \to \mu(K)$, hence without loss of generality we can suppose that the net μ_α is bounded, i.e. there is $M > 0$ such that $\mu_\alpha(K) \le M$ for each α. For every α let $B_\alpha \subset A$ be finite such that $\mu_\alpha(B_\alpha) \ge s$ and for each $b \in B_\alpha$ we have $\mu(\{b\}) \ge p$. Since each B_α has cardinality at most $\frac{M}{p}$, we can suppose (by passing to a subnet) that all B_α have the same cardinality, say n. Hence we can write $B_\alpha = \{b^1_\alpha, \dots, b^n_\alpha\}$. Since K is compact, each net has a converging subnet, so we can suppose that for each $k = 1, \dots, n$ the net b^k_α converges to some $b^k \in K$. Moreover, $b^k \in A$ for A is closed. Put $B = \{b^1, \dots, b^n\}$. At first let us notice that $\mu\left(b^k\right) \ge p$. Suppose not, which means $\mu\left(b^k\right) < p$. By the regularity of μ there is U, an open neighborhood of b^k such that $\mu(U) < p$. By the regularity of K there is V open neighborhood of b^k such that $\overline{V} \subset U$. Then $\mu\left(\overline{V}\right) < p$ and hence the set $W = \left\{\nu \in \mathcal{M}_+(K) \mid \nu\left(\overline{V}\right) < p\right\}$ is a neighborhood of μ, therefore there is α_0 such that for $\alpha > \alpha_0$ we have $\mu_\alpha \in W$. But since b^k_α converges to b^k, there is $\alpha_1 > \alpha_0$ such that for $\alpha > \alpha_1$ we have $b^k_\alpha \in V$. So, whenever $\alpha > \alpha_1$ then $\mu_\alpha\left(b^k_\alpha\right) \le \mu_\alpha(V) < p$, a contradiction. Thus $\mu\left(b^k\right) \ge p$. In a similar way we can see that $\mu(B) \ge s$. It follows that $F^s_{A,p}$ is closed, and by (1) that $G^s_{A,p}$ is $\mathcal{F}_\sigma$.

(3) Let $t > s$ be rational and $\mu \in G^t_{A,p} \setminus G^{t-s}_{B,p}$. Put $C = \{x \in A \mid \mu(\{x\}) \ge p\}$. Then $\mu(C) > t$ and $\mu(C \cap B) \le t-s$, so $\mu(C \setminus B) = \mu(C) - \mu(C\cap B) > t-(t-s) = s$, hence $\mu \in G^s_{A\setminus B,p}$, which proves the inclusion "$\supset$". To see the other one let $\mu \in G^s_{A\setminus B,p}$. Put $C = \{x \in A \mid \mu(\{x\}) \ge p\}$. Then $\mu(C \setminus B) > s$, and thus $\mu(C) > s+\mu(C\cap B)$. Choose a rational number t such that $\mu(C) > t > s+\mu(C\cap B)$. Then $\mu \in G^t_{A,p}$ but $\mu \notin G^{t-s}_{A\cap B,p}$ (since $\mu(C \cap B) < t - s$), which completes the proof.

(4) The inclusion "$\supset$" is obvious, let us show the other one. Let $\mu \in G^s_{\bigcup \mathcal{A},p}$. By the definition there is $B \subset \bigcup \mathcal{A}$ finite such that $\mu(B) > s$ and $\mu(b) \ge p$ for all $b \in B$. Since B is finite and $\mathcal{A}$ closed to finite unions there is $A \in \mathcal{A}$ such that $B \subset A$. It follows that $\mu \in G^s_{A,p}$, which yields the assertion. The same proof works evidently for $F^s_{A,p}$ too.

(5) The inclusion "$\subset$" is obvious. Let us consider the inverse one. Choose $\mu \in \bigcap\left\{G^s_{A,p} \mid A \in \mathcal{A}\right\}$. For $A \in \mathcal{A}$ put $B_A = \{x \in A \mid \mu(\{x\}) \ge p\}$. Clearly $B_{A_1 \cap A_2} = B_{A_1} \cap B_{A_2}$ and $\mu(B_A) > s$ for every $A, A_1, A_2 \in \mathcal{A}$. We shall prove that for some A we have $B_A \subset \bigcap \mathcal{A}$. Suppose not. Choose $A_1 \in \mathcal{A}$ arbitrary. By the assumption we can construct $A_n \in \mathcal{A}$ such that for every n we have $B_{A_n} \setminus A_{n+1} \ne \emptyset$. But this is a contradiction, since B_{A_1} is finite and we cannot have an infinite strictly decreasing sequence of finite sets. The same proof works clearly also for $F^s_{A,p}$. $\square$

Lemma 2. *Let $p, s > 0$. Then the following hold.*

(i) If $A \subset K$ is Borel then $G^s_{A,p}$ and $F^s_{A,p}$ are Borel sets.

(ii) If $A \subset K$ is Suslin-$\mathcal{F}$ then $G^s_{A,p}$ and $F^s_{A,p}$ are Suslin-$\mathcal{F}$, too.

(iii) If $A \subset K$ is Suslin-$\mathcal{B}$ then $G^s_{A,p}$ and $F^s_{A,p}$ are Suslin-$\mathcal{B}$, too.

(iv) If $A \subset K$ is co-Suslin-$\mathcal{F}$ (or co-Suslin-$\mathcal{B}$) then $G^s_{A,p}$ and $F^s_{A,p}$ are co-Suslin-$\mathcal{F}$ (or co-Suslin-$\mathcal{B}$), too.

Proof. (i) The family $\mathcal{M} = \left\{A \subset K \mid G^s_{A,p}, F^s_{A,p} \text{ are Borel for every } p, s > 0\right\}$ contains closed sets (by Lemma 1(2)), is closed with respect to complements (Lemma 1(3)), and with respect to monotone countable unions and intersections (it follows easily from Lemma 1(4),(5)), so it is standard to check that $\mathcal{M}$ contains all Borel sets.

(ii) and (iii) Suppose that $A = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} A_{\alpha\restriction n}$ where A_s are closed (resp. Borel). To clarify the notation let us remark that for an infinite sequence of natural numbers α and a natural number n we denote by $\alpha \restriction n$ the finite sequence formed by the first n members of α. We can suppose that this Suslin operation is monotone, i.e. that $A_s \subset A_t$ if t is a beginning of s. We can write

$$A = \bigcup_{\alpha_1,\dots,\alpha_k\in\mathbb{N}^{\mathbb{N}},k\in\mathbb{N}} \left(\bigcup_{j=1}^{k} \bigcap_{n\in\mathbb{N}} A_{\alpha_j\restriction n} \right) = \bigcup_{\alpha_1,\dots,\alpha_k\in\mathbb{N}^{\mathbb{N}},k\in\mathbb{N}} \left(\bigcap_{n\in\mathbb{N}} \bigcup_{j=1}^{k} A_{\alpha_j\restriction n} \right),$$

so by Lemma 1(4),(5) we get

$$G^s_{A,p} = \bigcup_{\alpha_1,\dots,\alpha_k\in\mathbb{N}^{\mathbb{N}},k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} G^s_{\bigcup_{j=1}^k A_{\alpha_j\restriction n},p} = \bigcup_{k\in\mathbb{N}} \bigcup_{(\alpha_1,\dots,\alpha_k)\in(\mathbb{N}^{\mathbb{N}})^k} \bigcap_{n\in\mathbb{N}} G^s_{\bigcup_{j=1}^k A_{\alpha_j\restriction n},p}.$$

Each of the sets $\bigcup_{j=1}^k A_{\alpha_j\restriction n}$ is closed (resp. Borel), so $G^s_{\bigcup_{j=1}^k A_{\alpha_j\restriction n},p}$ is $\mathcal{F}_\sigma$ (resp. Borel) by Lemma 1(2) (resp. by (i)). It follows that $G^s_{A,p}$ is a countable union of results of Suslin operation on $\mathcal{F}_\sigma$-sets (resp. on Borel sets), hence $G^s_{A,p}$ is Suslin-$\mathcal{F}$ (resp. Suslin-$\mathcal{B}$). Similarly for $F^s_{A,p}$.

(iv) Let "Suslin" mean either Suslin-$\mathcal{F}$ or Suslin-$\mathcal{B}$. If A is co-Suslin then $B = K \setminus A$ is Suslin. By Lemma 1(3) we have $G^s_{A,p} = \bigcup_{t>s,t\in\mathbb{Q}} \left(G^t_{K,p} \setminus G^{t-s}_{B,p} \right)$. The sets $G^t_{K_p}$ are $\mathcal{F}_\sigma$ by Lemma 1(2), the sets $G^{t-s}_{B,p}$ are Suslin by (iii), so the differences $G^t_{K,p} \setminus G^{t-s}_{B,p}$ are co-Suslin, and so is $G^s_{A,p}$. Similarly for $F^s_{A,p}$. ☐

Proof of Theorem 2. We have

$$\{\mu \in \mathcal{M}_+(K) \mid (\exists a \in A)(\mu(\{a\}) > c)\} = \bigcup_{s>0,s\in\mathbb{Q}} \bigcup_{n\in\mathbb{N}} F^s_{A,c+\frac{1}{n}}, \quad \text{and}$$

$$\{\mu \in \mathcal{M}_+(K) \mid (\exists a \in A)(\mu(\{a\}) \geq c)\} = \bigcup_{s>0,s\in\mathbb{Q}} F^s_{A,c}.$$

Hence the theorem follows immediately from Lemma 2. ☐

4. On structure of the space of Radon measures on K_B.

In this section we will consider a fixed compact perfect set $K \subset \mathbb{R}$ and a fixed set $B \subset K^d$ (K^d is the set of all both-side accumulation points of K). We will study the properties of the space K_B and of measures on that space. We adopt the notation of [Ka3], which forms the previous chapter of this thesis. The main result of this section is the following example. (The consistency of the used axioms with ZFC was proved in [ST], see also [Ka3].)

Example. *Suppose* MA & ¬CH & $\aleph_1 = \aleph_1^L$. *Let $B \subset K^d$ have cardinality $\aleph_1$. Then K_B is Stegall, non-fragmentable and every compact ccc subset of $\mathcal{M}_+(K_B)$ contains a dense completely metrizable subset.*

In view of Theorem A this is a partial result in the attempt to find an example of nonfragmentable compact space such that the dual unit ball of the space of continuous functions belongs to the Stegall class (i.e. such that the space of continuous functions is Stegall space whose dual is not fragmentable). But we do not know if, in our example, at least all compact subsets of $\mathcal{M}_+(K_B)$ contain a dense completely metrizable subspace. We know that this holds for ccc compact sets and also for a slightly more general class (see Proposition 8) but this does not cover all compact sets as it is witnessed by the following proposition.

Proposition 4. *The space $P(K_B)$ contains a homeomorphic copy of $D^{\mathbb{N}}$ where D is the discrete space of the same cardinality as B.*

Proof. By Proposition 2 the space $P(K_B)$ contains a homeomorphic copy of ${K_B}^{\mathbb{N}}$. Clearly ${K_B}^{\mathbb{N}}$ is homeomorphic with $(K_B \times K_B)^{\mathbb{N}}$. The set $D = \{((b,0),(b,1)) \mid b \in B\}$ is relatively discrete in $K_B \times K_B$ and has the same cardinality as B. So $P(K_B)$ contains a copy of $D^{\mathbb{N}}$. $\square$

If B is uncountable and we take H to be the closure in $\mathcal{M}_+(K_B)$ of the copy of $D^{\mathbb{N}}$ (whose existence was proved in the previous proposition), then H is compact and "locally non-ccc". However this H contains a dense completely metrizable subspace.

Now we will prove a sequence of propositions which leads to the proof of Example. Most of these propositions are stated in a more general setting than they are really needed here.

Proposition 5. *Let $A \subset B$ be arbitrary. Let $F : K_B \to K_A$ be the canonic surjection (as in* [Ka3, Proposition 6]*). Then the following hold.*

(1) If $\nu \in \mathcal{M}(K_A)$ is such that $\nu(\{x\}) = 0$ for every $x \in (B \setminus A) \times \{0\}$, then the formula $\mu(M) = \nu(F(M))$ for $M \subset K_B$ Borel defines a measure $\mu \in \mathcal{M}(K_B)$.

(2) For every $\mu \in \mathcal{M}(K_B)$ there exists a unique pair of measures $\nu_1 \in \mathcal{M}(K_A)$ and $\nu_2 \in \mathcal{M}(K_B)$ such that $\nu_1(\{x\}) = 0$ for every $x \in (B \setminus A) \times \{0\}$ and ν_2 is supported by a countable subset of $(B \setminus A) \times \{0,1\}$ and for any $M \subset K_B$ Borel we have $\mu(M) = \nu_1(F(M)) + \nu_2(M)$.

Proof. (1) By [Ka3, Proposition 6(3)], $F(M)$ is Borel whenever M is Borel, so we can define $\mu(M) = \nu(F(M))$ for every Borel set $M \subset K_B$. If M_n is a sequence of disjoint Borel subsets of K_B, it follows from [Ka3, Proposition 6(3)] that $F(M_k) \cap F(M_l)$ is an at most countable subset of $(B \setminus A) \times \{0\}$ for any pair k, l of distinct integers, so it is a ν-null set. Now it easily follows that μ is σ-additive; hence it is a Borel measure. Moreover, K_B is compact and it can be easily seen that each open subset of K_B is $\mathcal{F}_\sigma$, therefore by [GP, Theorem 11.15] any finite Borel measure on K_B is a Radon one, which completes the proof.

(2) If $\mu \in \mathcal{M}(K_B)$ then we put $\nu_2 = \sum\{\mu(\{x\})\delta_x \mid x \in (B \setminus A) \times \{0,1\}\}$ and $\nu_1 = F(\mu - \nu_2)$. The pair ν_1, ν_2 is clearly the unique one which satisfies the required conditions. $\square$

Lemma 3. *Let $A \subset B$ be arbitrary and $F : K_B \to K_A$ be the canonic surjection (as in* [Ka3, Proposition 6]*) and $\tilde{F}^\star$ have the same meaning as in Proposition 1. Put*

$$M = \{\mu \in \mathcal{M}_+(K_B) \mid (\forall t \in B \setminus A)\, (\mu(\{(t,0)\}) = \mu(\{(t,1)\}) = 0)\}.$$

Then $\tilde{F}^\star$ maps homeomorphically M onto the set

$$\{\nu \in \mathcal{M}_+(K_A) \mid (\forall t \in B \setminus A)\, (\nu(\{(t,0)\}) = 0)\}.$$

Proof. By Proposition 1 the map $\tilde{F}^\star$ is continuous. It follows from Proposition 5 that $\tilde{F}^\star$ is one-to-one on M and maps M onto the mentioned set. It remains to prove that $\tilde{F}^\star$ is open onto its image. To see it let $\mu \in M$ be arbitrary and $U = \{\nu \in \mathcal{M}_+(K_B) \mid \nu(K_B) < a, \nu(G_i) > b_i,\ i = 1, \dots, n\}$, where $G_1, \dots, G_n$ are open in K_B, be a neighborhood of μ. We will show that $\tilde{F}^\star(U \cap M)$ is relatively open in $\tilde{F}^\star(M)$. Let $C_i = \{(t, \varepsilon) \in G_i \mid t \in B \setminus A \,\&\, (t, 1-\varepsilon) \notin G_i\}$ and $H_i = G_i \setminus C_i$. Then it is easy to see, by the definition of the topology of K_A, that C_i is countable, H_i open in K_B and $F(H_i)$ open in K_A. Moreover, for $\nu \in M$ we have $\nu(C_i) = 0$ and hence $\nu(G_i) = \nu(H_i)$, therefore $U \cap M = \tilde{U} \cap M$, where $\tilde{U} = \{\nu \in \mathcal{M}_+(K_B) \mid \nu(K_B) < a, \nu(H_i) > b_i,\ i = 1, \dots, n\}$. Moreover,

$$\tilde{F}^\star\left(\tilde{U}\right) = \{\nu \in \mathcal{M}_+(K_A) \mid \nu(K_A) < a, \nu\left(\tilde{F}^\star(H_i)\right) > b_i,\ i = 1, \dots, n\}$$

is open in $\mathcal{M}_+(K_A)$ and $\tilde{F}^\star\left(\tilde{U} \cap M\right) = \tilde{F}^\star\left(\tilde{U}\right) \cap \tilde{F}^\star(M)$. As for the last equality, the inclusion "$\subset$" is obvious. To see the inverse one let $\nu \in \tilde{F}^\star\left(\tilde{U}\right) \cap \tilde{F}^\star(M)$. Since $\nu \in \tilde{F}^\star(M)$ we have $\nu(\{(b, 0)\}) = 0$ for $b \in B \setminus A$, so by Proposition 5(1) we can define a measure $\mu \in \mathcal{M}_+(K_B)$ by the formula $\mu(S) = \nu(F(S))$ for $S \subset K_B$ Borel. Now, clearly $\mu \in \tilde{U} \cap M$, which completes the proof. $\square$

In the next proposition we evaluate the weight of any subset of $\mathcal{M}_+(K_B)$. We will use later a particular case, namely the characterization of second countable subsets of $\mathcal{M}_+(K_B)$.

Proposition 6. *Let $N \subset \mathcal{M}_+(K_B)$ be arbitrary. Then*

$$w(N) + \aleph_0 = nw(N) + \aleph_0 = \operatorname{card}\{t \in B \mid (\exists \mu \in N)\,(\mu(\{(t, 0), (t, 1)\}) > 0)\} + \aleph_0,$$

where $w(N)$ denotes the weight of N (in the $w^\star$ topology), and $nw(N)$ is the "netweight" of N (i.e., minimal cardinality of a network of N).

Proof. The inequality $nw(N) \leq w(N)$ is obvious.

Next we will show that

$$w(N) + \aleph_0 \leq \operatorname{card}\{t \in B \mid (\exists \mu \in N)\,(\mu(\{(t, 0), (t, 1)\}) > 0)\} + \aleph_0.$$

Put $A = \{t \in B \mid (\exists \mu \in N)\,(\mu(\{(t, 0), (t, 1)\}) > 0)\}$. Let F and $\tilde{F}^\star$ have the same meaning as in Lemma 3. So, by Lemma 3 the map $\tilde{F}^\star$ is a homeomorphism of N onto its image, so it is enough to show that $w(\mathcal{M}_+(K_A)) \leq \operatorname{card} A + \aleph_0$. But this follows from Proposition 3(1) since the weight of K_A is clearly $\leq \operatorname{card} A + \aleph_0$.

It remains to show that

$$nw(N) + \aleph_0 \geq \operatorname{card}\{t \in B \mid (\exists \mu \in N)\,(\mu(\{(t, 0), (t, 1)\}) > 0)\} + \aleph_0.$$

Let $\mathcal{N}$ be a network of N. Put

$$\kappa = \operatorname{card} \mathcal{N} + \aleph_0 \text{ and } \tau = \operatorname{card}\{t \in B \mid (\exists \mu \in N)\,(\mu(\{(t, 0), (t, 1)\}) > 0)\} + \aleph_0.$$

Suppose that $\tau > \kappa$, which means $\tau \geq \kappa^+$. In this case τ is necessarily uncountable (and hence $\tau = \operatorname{card}\{t \in B \mid (\exists \mu \in N)(\mu(\{(t,0),(t,1)\}) > 0)\}$). It follows that there is a positive rational number p such that

$$\operatorname{card}\left\{t \in B \mid (\exists \mu \in N)\left(\mu(\{(t,0),(t,1)\}) \in \left(p, \frac{3}{2}p\right)\right)\right\} \geq \kappa^+.$$

Without loss of generality we can suppose that $\operatorname{card} C \geq \kappa^+$, where

$$C = \left\{t \in B \mid (\exists \mu \in N)\left(\mu(\{(t,0),(t,1)\}) \in \left(p, \frac{3}{2}p\right) \& \mu(\{(t,1)\}) \leq \mu(\{(t,0)\})\right)\right\}.$$

Otherwise the set with the inequality "$\leq$" replaced by "$\geq$" would have cardinality at least κ^+ and the rest of the proof would be analogous as in our case.

Let $t \in C$ be arbitrary. Choose $\mu_t \in N$ satisfying the above condition. Then $\mu_t(\{(t,1)\}) < \frac{3}{4}p$, hence by regularity of μ_t there is $\varepsilon_t > 0$ such that

$$\mu_t(\{(t,1)\} \cup ((t, t+\varepsilon_t] \times \{0,1\}) \cap K_B) < \frac{3}{4}p.$$

Now,

$$U_t = \left\{\mu \in N \mid \mu(\{(t,1)\} \cup ((t, t+\varepsilon_t] \times \{0,1\}) \cap K_B) < \frac{3}{4}p\right\}$$

is a neighborhood of μ_t in N, so there is $N_t \in \mathcal{N}$ such that $\mu_t \in N_t \subset U_t$. Next we will show that the map $t \mapsto N_t$ is countable-to-one, and therefore $\operatorname{card}\mathcal{N} \geq \kappa^+$, which will be a contradiction. Choose $M \in \mathcal{N}$ and put $A = \{t \in C \mid N_t = M\}$. Let us show that every point of A is right-isolated, namely if $t, s \in A$, $t < s$ then $s > t+\varepsilon_t$. Of course, if $t < s \leq t+\varepsilon_t$ then $\mu_s(\{(t,1)\} \cup ((t, t+\varepsilon_t] \times \{0,1\}) \cap K_B) \geq \mu_s(\{(s,0),(s,1)\}) > p > \frac{3}{4}p$, hence $\mu_s \notin U_t$, so $N_t \neq N_s$. Thus necessarily A is countable and we are done. $\square$

In the following proposition we in particular evaluate the Suslin number of some subsets of $\mathcal{M}_+(K_B)$ and characterize compact subsets of $\mathcal{M}_+(K_B)$ which contain a dense completely metrizable subspace within ccc subsets.

Proposition 7. *(1) The space $\mathcal{M}_+(K_B)$ is first countable and has cardinality $2^{\aleph_0}$.*

(2) Let $H \subset \mathcal{M}_+(K_B)$ be an arbitrary compact subset. Put

$$A = \{b \in B \mid (\exists \mu \in H)(\mu(\{(b,0),(b,1)\}) > 0)\},$$
$$C = \{b \in B \mid \{\mu \in H \mid \mu(\{(b,0),(b,1)\}) > 0\} \text{ has nonempty interior in } H\},$$
$$L = \{\mu \in H \mid (\exists b \in A \setminus C)(\mu(\{(b,0),(b,1)\}) > 0)\}.$$

(a) If L has empty interior in H then $c(H) + \aleph_0 = \operatorname{card} C + \aleph_0$ (where $c(H)$ denotes the Suslin number of H).

(b) If relatively open subsets of H which satisfy ccc form a pseudobase of H, then H contains a dense completely metrizable subspace if and only if L

is meager in H. In this case H contains a dense locally separable completely metrizable subspace.

Proof. (1) Let $\mu \in \mathcal{M}_+(K_B)$. Put $S = \{b \in B \mid \mu(\{(b,0),(b,1)\}) > 0\}$. Then S is countable and the family

$$\mathcal{P} = \{(p,q) \times \{0,1\} \cap K_B \mid p,q \in \mathbb{Q}\} \cup \{(s - \frac{1}{n}, s) \times \{0,1\} \cup \{(s,0)\} \cap K_B, \\ (s, s + \frac{1}{n}) \times \{0,1\} \cup \{(s,1)\} \cap K_B \mid s \in S\}$$

satisfies the condition required in Proposition 3(2). Moreover, $\mathcal{P}$ is countable, therefore $\mathcal{M}_+(K_B)$ is first countable.

Now, by Proposition 6 we have $w(\mathcal{M}_+(K_B)) \leq 2^{\aleph_0}$, hence there is a dense subset D of $\mathcal{M}_+(K_B)$ of cardinality $\leq 2^{\aleph_0}$. Since every point of $\mathcal{M}_+(K_B)$ is the limit of a sequence in D, it follows that $\operatorname{card} \mathcal{M}_+(K_B) \leq 2^{\aleph_0}$. The inverse inequality is obvious since $\mathcal{M}_+(K_B)$ contains a copy of K_B.

(2a) If L has empty interior in H then $H \setminus L$ is dense in H, and therefore $c(H) = c(H \setminus L)$. By Proposition 6 we have $w(H \setminus L) + \aleph_0 = \operatorname{card} C + \aleph_0$, hence $c(H) = c(H \setminus L) \leq w(H \setminus L) \leq \operatorname{card} C + \aleph_0$. It remains to show that $\operatorname{card} C \leq c(H) + \aleph_0$. Suppose not. Put $\kappa = c(H) + \aleph_0$, $\tau = \operatorname{card} C$, and assume that $\tau \geq \kappa^+$. Now, H is compact, and hence is bounded, let $M > 0$ be such that for every $\mu \in H$ we have $\mu(K_B) < M$. Put $C_n = \{b \in B \mid \{\mu \in H \mid \mu(\{(b,0),(b,1)\}) \geq \frac{M}{n})\}$ has nonempty interior in $H\}$. Then $C = \bigcup_{n \in \mathbb{N}} C_n$. The inclusion $\supset$ is obvious, to see the inverse choose $b \in C$. Then

$$\{\mu \in H \mid \mu(\{(b,0),(b,1)\}) > 0\} = \bigcup_{n \in \mathbb{N}} \{\mu \in H \mid \mu(\{(b,0),(b,1)\}) \geq \frac{M}{n}\},$$

the set on the left-hand side has nonempty interior, and hence is nonmeager, and each of the sets on the right-hand side is closed (by the definition of the $w^\star$ topology), so one of them has nonempty interior, which means that $b \in C_n$ for some n. It follows that for some n we have $\operatorname{card} C_n \geq \kappa^+$. For $b \in C_n$ put $U_b = \operatorname{int}_H\{\mu \in H \mid \mu(\{(b,0),(b,1)\}) \geq \frac{M}{n}\}$. Then $(U_b \mid b \in C_n)$ is a family of nonempty relatively open subsets of H which is "point-$\leq n$", i.e. every $\mu \in H$ belongs to at most n different U_b's. Choose minimal m such that there is a family of nonempty relatively open subsets of H which is "point-$\leq m$" and has cardinality at least κ^+. Such an m exists ($m \leq n$ for n chosen above) and $m > 1$ (since $c(H) \leq \kappa$). Let $\mathcal{U}$ be such a family. Put $\mathcal{Z} = \{\mathcal{V} \subset \mathcal{U} \mid \operatorname{card} \mathcal{V} = m \ \& \ \bigcap \mathcal{V} \neq \emptyset\}$. For $\mathcal{V}, \mathcal{V}' \in \mathcal{Z}$ different we have $\bigcap \mathcal{V} \cap \bigcap \mathcal{V}' = \emptyset$ (since $\mathcal{U}$ is "point-$\leq m$"), hence $\operatorname{card} \mathcal{Z} \leq \kappa$ (for $c(H) \leq \kappa$) and also $\operatorname{card} \bigcup \mathcal{Z} \leq \kappa$. Then $\operatorname{card}(\mathcal{U} \setminus \bigcup \mathcal{Z}) \geq \kappa^+$ and $\mathcal{U} \setminus \bigcup \mathcal{Z}$ is "point-$\leq (m-1)$", which is a contradiction with the minimality of m.

(b) Now suppose that the relatively open subsets of H which satisfy ccc form a pseudobase of H. If L is meager in H then $H \setminus L$ is residual. Let U be the union of all relatively open ccc subsets of H. By the assumption U is open dense in H, so $U \cap (H \setminus L)$ is residual in H. Let G be a dense $\mathcal{G}_\delta$ subset of H contained in $U \cap (H \setminus L)$. Choose $\mu \in G$ arbitrary. Then there is V, a neighborhood of μ in H, which is ccc. So, by the previous paragraph, the set $\{b \in C \mid (\exists \mu \in V \cap G)(\mu(\{(b,0),(b,1)\}) > 0)\}$ is countable. And since, by the construction, $\{b \in B \mid (\exists \mu \in V \cap G)(\mu(\{(b,0),(b,1)\}) > 0)\} \subset C$ we get (by Proposition 6) that $V \cap G$

has countable basis, and thus it is a separable completely metrizable space (for it is obviously Čech complete). It follows that each point of G has a neighborhood which is separable and completely metrizable. Finally, let Y be the union of a maximal disjoint family of relatively open subsets of G each member of which is a separable completely metrizable space. Then clearly Y is dense open subset of G (and hence dense $\mathcal{G}_\delta$ in H) which is completely metrizable and locally separable.

Conversely, suppose that $G \subset H$ is a dense completely metrizable space but L is non-meager in H. By the Banach localization principle there is U, a nonempty relatively open subset of H such that $L \cap U$ is a dense Baire subspace of U. By the assumption there is $V \subset U$ nonempty relatively open such that V satisfies ccc. Then $L \cap V$ is a dense Baire subspace of V, and hence $G \cap L \cap V$ is non-meager in H and dense in V. Now, V is ccc, and hence $G \cap V$ is ccc. Since G is metrizable, $G \cap V$ has countable basis. By Proposition 6 the set $\{b \in B \mid (\exists \mu \in G \cap V)(\mu(\{(b,0),(b,1)\}) > 0)\}$ is countable. Therefore there is $E \subset A \setminus C$ countable such that

$$G \cap L \cap V \subset \bigcup_{e \in E} \{\mu \in H \mid \mu(\{(e,0),(e,1)\}) > 0\}.$$

The sets on the right-hand side have empty interior in H (for $E \cap C = \emptyset$), are $\mathcal{F}_\sigma$ and hence are meager. It follows that $G \cap L \cap V$ is meager, a contradiction. □

Proposition 8. *Suppose that every subset of B is coanalytic. Let $H \subset \mathcal{M}_+(K_B)$ be compact and A, C, L have the same meaning as in Proposition 7. Then L is meager in H. In particular, if relatively open subsets of H which satisfy ccc form a pseudobase of H, then H contains a dense completely metrizable (locally separable) subspace.*

Proof. For $b \in A \setminus C$ and $n \in \mathbb{N}$ put $L_b^n = \{\mu \in H \mid (\exists i \in \{0,1\})(\mu(\{(b,i)\}) \geq \frac{1}{n}\}$. Each L_b^n has empty interior in H, is $\mathcal{F}_\sigma$ (by Lemma 1(2)), and hence is meager in H. Moreover, for fixed n, the family $(L_b^n \mid b \in A \setminus C)$ is point-finite and the union of each subfamily is co-Suslin (by Theorem 2), in particular has the restricted Baire property. By Proposition 7(1) we have $\operatorname{card} H \leq 2^{\aleph_0}$. Therefore, by Theorem 1(ii) $\bigcup_{b \in A \setminus C} L_b^n$ is meager in H for every n, so $L = \bigcup_{n \in \mathbb{N}} \bigcup_{b \in A \setminus C} L_b^n$ is meager in H, and the rest follows by Proposition 7. □

Proof of Example. By [MS, Theorem 3.2], under the assumptions of Example, every subset of $\mathbb{R}$ of cardinality $\aleph_1$ is coanalytic. In particular, every subset of B is coanalytic. Moreover, by [MS, p.162] each subset of B is relatively $\mathcal{F}_\sigma$ (i.e., it is a $\mathcal{Q}$-set). By [Ka3] the space K_B is Stegall and non-fragmentable (in fact, we do not need the fact that B is a $\mathcal{Q}$-set, if we modify a bit Proposition 7(b) in [Ka3]). The rest follows from Proposition 8. □

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