4.8. **Concave and quasiconcave functions.**

Definition. Let $M \subset \mathbf{R}^n$. We say that M is *convex*, if we have

$$\forall \boldsymbol{x}, \boldsymbol{y} \in M\ \forall t \in \langle 0, 1\rangle \colon t\boldsymbol{x} + (1-t)\boldsymbol{y} \in M.$$

Definition. Let $M \subset \mathbf{R}^n$ be a convex set and a function f be defined on M. We say that f is

- *concave on M*, if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M\ \forall t \in \langle 0, 1\rangle \colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}),$$

- *strictly concave on M*, if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}\ \forall t \in (0, 1) \colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}).$$

Remark. Let $M \subset \mathbf{R}^n$ be a convex set and $f : M \to \mathbf{R}$ a function. The following assertions are equivalent:

(i) f is concave.
(ii) The restriction of f to any segment in M is concave.
(iii) For any $a, b \in M$ the function $t \mapsto f(a + t(b-a))$ is concave on $\langle 0, 1\rangle$.

A similar equivalence is valid for strictly concave functions.

Theorem 4.22. *Let a function f be concave on an open convex set $G \subset \mathbf{R}^n$. Then f is continuous on G.*

Theorem 4.23. *Let a function f be concave on a convex set $M \subset \mathbf{R}^n$. Then for each $\alpha \in \mathbf{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.*

Theorem 4.24 (characterization of concave functions of the class $\mathcal{C}^1$)**.** *Let $G \subset \mathbf{R}^n$ be a convex open set and $f \in \mathcal{C}^1(G)$. Then the function f is convex on G if and only if we have*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G \colon f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Corollary 4.25. *Let $G \subset \mathbf{R}^n$ be a convex open set and $f \in \mathcal{C}^1(G)$ be concave on G. If a point $\boldsymbol{a} \in G$ is a stationary point of f, then $\boldsymbol{a}$ is a point of maximum of f with respect to G.*

Theorem 4.26 (characterization of strictly concave functions of the class $\mathcal{C}^1$)**.** *Let $G \subset \mathbf{R}^n$ be a convex open set and $f \in \mathcal{C}^1(G)$. Then the function f is strictly concave on G if and only if we have*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G, \boldsymbol{x} \neq \boldsymbol{y} \colon f(\boldsymbol{y}) < f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Definition. Let $M \subset \mathbf{R}^n$ be a convex set and f be defined on M. We say that f is

- *quasiconcave on M*, if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M\ \forall t \in [0, 1] \colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},$$

- *strictly quasiconcave on M*, if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b},\ \forall t \in (0, 1) \colon f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\}.$$

Remark. Let $M \subset \mathbf{R}^n$ be a convex set and f be a function defined on M.

- Let f be concave on M. Then f is quasiconcave on M.
- Let f be strictly concave on M. Then f is strictly quasiconcave on M.

Remark. Let $M \subset \mathbf{R}^n$ be a convex set and $f : M \to \mathbf{R}$ a function. The following assertions are equivalent:

(i) f is quasiconcave.
(ii) The restriction of f to any segment in M is quasiconcave.
(iii) For any $a, b \in M$ the function $t \mapsto f(a + t(b - a))$ is quasiconcave on $\langle 0, 1 \rangle$.

A similar equivalence is valid for strictly quasiconcave functions.

Remark. Let $I \subset \mathbf{R}$ be an interval and $f : I \to \mathbf{R}$ is a function.

- The function f is quasiconcave on I if and only if one of the following conditions is fulfilled
  (a) f is non-decreasing on I.
  (b) f is non-increasing on I.
  (c) There is $x \in I$ such that f is non-decreasing on $I \cap (-\infty, a\rangle$ and non-increasing on $I \cap \langle a, +\infty)$.
- The function f is strictly quasiconcave on I if and only if one of the following conditions is fulfilled
  (a) f is strictly decreasing on I.
  (b) f is strictly increasing on I.
  (c) There is $x \in I$ such that f is trictly increasing on $I \cap (-\infty, a\rangle$ and strictly decreasing on $I \cap \langle a, +\infty)$.

Theorem 4.27 (characterization of quasiconcave functions via level sets)**.** *Let $M \subset \mathbf{R}^n$ be a convex set and f be defined on M. The function f is quasiconcave on M if and only if for each $\alpha \in \mathbf{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.*

Theorem 4.28 (on uniqueness of extremum)**.** *Let f be a strictly quasiconcave function on a convex set $M \subset \mathbf{R}^n$. Then there exists at most one point of maximum of f.*

Corollary 4.29. *Let $M \subset \mathbf{R}^n$ be a convex, bounded, closed and nonempty set. Let f be continuous and strictly quasiconcave function on M. Then f attains its maximum on M in a unique point.*