4.6. **Implicit function theorem.**

Theorem 4.18 (implicit function theorem – basic version)**.** *Let $G \subset \mathbf{R}^{n+1}$ be an open set, $F\colon G \to \mathbf{R}$, $\tilde{\boldsymbol{x}} \in \mathbf{R}^n$, $\tilde{y} \in \mathbf{R}$, $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that*

(1) $F \in \mathcal{C}^1(G)$,
(2) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,
(3) $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

Then there exist a neighborhood $U \subset \mathbf{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighborhood $V \subset \mathbf{R}$ of the point $\tilde{y}$ such that for each $\boldsymbol{x} \in U$ there exists unique $y \in V$ with the property $F(\boldsymbol{x}, y) = 0$. If we denote this y by $\varphi(\boldsymbol{x})$, then the resulting function φ is in $\mathcal{C}^1(U)$ and

$$\frac{\partial \varphi}{\partial x_j}(\boldsymbol{x}) = -\frac{\frac{\partial F}{\partial x_j}(\boldsymbol{x}, \varphi(\boldsymbol{x}))}{\frac{\partial F}{\partial y}(\boldsymbol{x}, \varphi(\boldsymbol{x}))} \quad \text{for } \boldsymbol{x} \in U,\ j \in \{1, \dots, n\}.$$

If, moreover, $F \in \mathcal{C}^k(G)$ for some $k \in \mathbf{N} \cup \{\infty\}$, then also $\varphi \in \mathcal{C}^k(U)$

Theorem 4.19 (implicit function theorem – advanced version)**.** *Let $m, n \in \mathbf{N}$, $k \in \mathbf{N} \cup \{\infty\}$, $G \subset \mathbf{R}^{n+m}$ be an open set, $F_j\colon G \to \mathbf{R}$ for $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbf{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbf{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. Suppose that*

(1) $F_j \in \mathcal{C}^k(G)$ *for each* $j \in \{1, \dots, m\}$,
(2) $F_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = 0$ *for each* $j \in \{1, \dots, m\}$,
(3) $\begin{vmatrix} \frac{\partial F_1}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_1}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_m}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \end{vmatrix} \neq 0$.

Then there exist a neighborhood $U \subset \mathbf{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighborhood $V \subset \mathbf{R}^m$ of the point $\tilde{\boldsymbol{y}}$ such that for each $\boldsymbol{x} \in U$ there exists unique $\boldsymbol{y} \in V$ with the property $F_j(\boldsymbol{x}, \boldsymbol{y}) = 0$ for each $j \in \{1, \dots, m\}$. If we denote coordinates of this $\boldsymbol{y}$ by $\varphi_j(\boldsymbol{x})$, $j = 1, \dots, m$, then the resulting functions φ_j are in $\mathcal{C}^k(U)$.

Remark. The symbol in the condition (3) of Theorem 4.19 is called *determinant*. The definition will presented in the next chapter.

For $m = 1$ we have $\begin{vmatrix} a \end{vmatrix} = a$, $a \in \mathbf{R}$.

For $m = 2$ we have $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, $a, b, c, d \in \mathbf{R}$.