6.3. **Integration of rational functions.**

Definition. *Rational function* is a ratio of two polynomials, where the polynomial in denominator is not identically zero.

Theorem 6.13 (decomposition to partial fractions)**.** *Let P, Q be polynomial functions with real coefficients such that*

(i) *degree of P is strictly smaller than degree of Q,*
(ii) $Q(x) = a_n(x-x_1)^{p_1}\dots(x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1}\dots(x^2+\alpha_l x+\beta_l)^{q_l}$*,*
(iii) $a_n,\ x_1,\dots x_k,\ \alpha_1,\dots,\alpha_l,\ \beta_1,\dots,\beta_l \in \mathbf{R},\ a_n \neq 0$*,*
(iv) $p_1,\dots,p_k,\ q_1,\dots,q_l \in \mathbf{N}$*,*
(v) *the polynomials $x-x_1,\ x-x_2,\dots,x-x_k,\ x^2+\alpha_1 x+\beta_1,\ \dots, x^2+\alpha_l x+\beta_l$ have no common root,*
(vi) *the polynomials $x^2+\alpha_1 x+\beta_1,\ \dots, x^2+\alpha_l x+\beta_l$ have no real root.*

Then there exist unique real numbers $A^1_1,\dots,A^1_{p_1},\dots,A^k_1,\dots,A^k_{p_k}, B^1_1,\ C^1_1,\dots,B^1_{q_1},C^1_{q_1},\dots,B^l_1,$ $C^l_1,\dots,B^l_{q_l},\ C^l_{q_l}$ such that we have

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{A^1_1}{(x-x_1)^{p_1}}+\dots+\frac{A^1_{p_1}}{(x-x_1)}\\ &+\dots+\frac{A^k_1}{(x-x_k)^{p_k}}+\dots+\frac{A^k_{p_k}}{x-x_k}\\ &+\frac{B^1_1 x+C^1_1}{(x^2+\alpha_1 x+\beta_1)^{q_1}}+\dots+\frac{B^1_{q_1}x+C^1_{q_1}}{x^2+\alpha_1 x+\beta_1}+\dots\\ &+\frac{B^l_1 x+C^l_1}{(x^2+\alpha_l x+\beta_l)^{q_l}}+\dots+\frac{B^l_{q_l}x+C^l_{q_l}}{x^2+\alpha_l x+\beta_l}.\end{aligned}$$

Remark. Any nonzero polynomial with real coefficients can be decomposed in the way described in the previous theorem for Q. In particular, if Q is a polynomial with real coefficients and $\lambda \in \mathbf{C}$ is a root of Q, then the complex conjugate $\overline{\lambda}$ is also a root of Q and has the same multiplicity as λ.

Remark. An antiderivative of a rational function $\frac{P(x)}{Q(x)}$ is computed as follows:

- Find polynomials R and Z such that degree of Z is smaller than degree of Q such that
$$\frac{P(x)}{Q(x)} = R(x) + \frac{Z(x)}{Q(x)}.$$
- Decompose $\frac{Z(x)}{Q(x)}$ as in the previous theorem.
- Compute antiderivative of R and of individual terms of the decomposition.